Homogenous super-Carrollian manifolds for the super Poincaré group

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Ongoing work with N. Boulanger and Y. Herfray

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Klein Geometry and homogeneous spaces

Homogeneous space

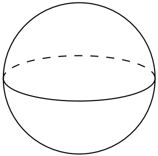
Space M with a transitive action of a Lie group G

"All points look the same"

$\longrightarrow M \simeq G/H$, where H is the stabilizer of one point $x \in G$

Examples :

- $\mathbb{S}^2 \simeq \frac{\mathrm{SO}(3)}{\mathrm{SO}(2)}$
- $\mathbb{M}^{1,3} \simeq \frac{\mathrm{ISO}(1,3)}{\mathrm{SO}(1,3)}$



Klein pair

The pair (G, H) is a Klein geometry

Example : Conformally compactified Minkowski

Conformally compactified Minkowski $\overline{\mathbb{M}}^{1,3}$ is a homogeneous space for the conformal group

$$\overline{\mathbb{M}}^{1,3} = \frac{\mathrm{SO}(2,4)}{\mathbb{R}^4 \rtimes (\mathbb{R} \times \mathrm{SO}(1,3))}$$

We can choose $\mathrm{ISO}(1,3) \subset \mathrm{SO}(2,4)$ and break the conformal invariance by imposing to stabilize the preferred degenerate direction, called null infinity tractor $I^I = \begin{bmatrix} 1 \\ 0^{AA'} \\ 0 \end{bmatrix}$

$\rightarrow$ Split of $\overline{\mathbb{M}}^{1,3}$ into orbits of Poincaré

Orbit decomposition

Three orbits (subspaces invariant under the action of Poincaré)

$$\overline{\mathbb{M}}^{1,3} = \mathbb{M}^{1,3} \sqcup \mathscr{I} \sqcup \{I\}$$

Because Poincaré acts transitively on each of these subspaces, they are homogeneous spaces for $\mathsf{ISO}(1,3)$:

$$\overline{\mathbb{M}}^{1,3} = \frac{\mathsf{ISO}(1,3)}{\mathsf{SO}(1,3)} \sqcup \frac{\mathsf{ISO}(1,3)}{\mathbb{R}^3 \rtimes (\mathbb{R} \times \mathsf{ISO}(2))} \sqcup \frac{\mathsf{ISO}(1,3)}{\mathsf{ISO}(1,3)}$$

Conformal Carrollian geometry on $\mathscr{I}$!
[Herfray20; Figueroa21]

Super Minkowski space

Goal : generalize this to the supersymmetric case

$$\begin{array}{ccc} \overline{\mathbb{M}}^4 & \rightarrow & \text{super compactified Minkowski space } \overline{\mathbb{M}}^{4|2\mathcal{N}} \\ \mathrm{SU}(2,2) \twoheadrightarrow \mathrm{SO}(2,4) & \rightarrow & \text{super conformal group } \mathrm{SU}(2,2|\mathcal{N}) \\ \mathrm{SU}(2,2) \circlearrowleft \overline{\mathbb{M}}^4 & \rightarrow & \mathrm{SU}(2,2|\mathcal{N}) \circlearrowleft \overline{\mathbb{M}}^{4|2\mathcal{N}} \end{array}$$

$\Rightarrow \overline{\mathbb{M}}^{4|2\mathcal{N}}$ is an homogeneous space for the superconformal group

e.g. [Manin97]

Question : Orbit decomposition of $\overline{\mathbb{M}}^{4|2\mathcal{N}}$ for super Poincaré group ?

Orbit decomposition : super case

Choice of a preferred super null direction $I^{\alpha b} = \begin{bmatrix} 1^{Ab} \\ 0_{A'}{}^{b} \\ 0^{Ib} \end{bmatrix}$

$\iff$ Choice of $\mathrm{ISO}(1,3|\mathcal{N}) \subset \mathrm{SU}(2,2|\mathcal{N})$

Result of the decomposition : more orbits !

$$\overline{\mathbb{M}}^{1,3|2\mathcal{N}} = \mathbb{M}^{1,3|2\mathcal{N}} \sqcup \mathscr{I}^{(3|\mathcal{N})} \sqcup \mathcal{O}_1 \sqcup \mathcal{O}_2 \sqcup \{I\}$$

Each of these is an homogeneous space for super Poincaré

$$\overline{\mathbb{M}}^{1,3|2\mathcal{N}} = \mathbb{M}^{1,3|2\mathcal{N}} \sqcup \mathscr{I}^{(3|\mathcal{N})} \sqcup \mathcal{O}_1^{(0|2\mathcal{N})} \sqcup \mathcal{O}_2^{(3|2\mathcal{N})} \sqcup \{I\}$$

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On $\mathbb{M}^{1,3|2\mathcal{N}} \simeq \frac{\mathrm{ISO}(1,3|\mathcal{N})}{\mathrm{SO}(1,3)\times\mathrm{SU}(\mathcal{N})}$ we find coordinates $(X_+^{AA'}, \theta^{A'}{}_I)$ such that one can write

$$X_+^{AA'} = X^{AA'} + \frac{i}{2}\theta^{A'}{}_I \bar{\theta}^A{}_{\bar{J}}\, h^{I\bar{J}},$$

for a real $X^{AA'}$: chiral coordinates appear naturally!

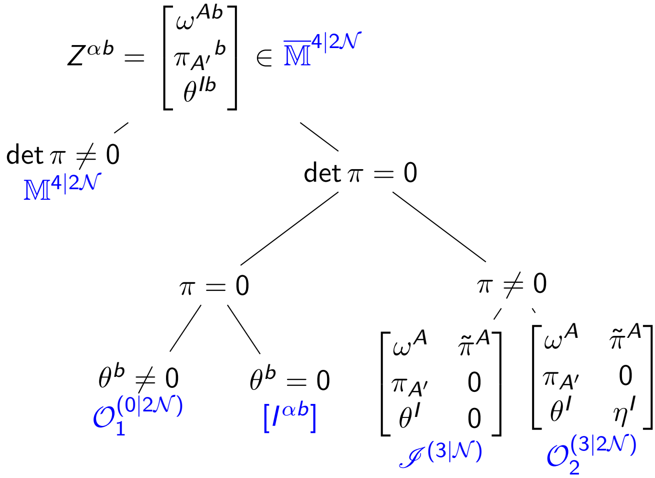
$$\overline{\mathbb{M}}^{1,3|2\mathcal{N}} = \mathbb{M}^{1,3|2\mathcal{N}} \sqcup \mathscr{I}^{(3|\mathcal{N})} \sqcup \mathcal{O}_1^{(0|2\mathcal{N})} \sqcup \mathcal{O}_2^{(3|2\mathcal{N})} \sqcup \{I\}$$

On $\mathscr{I}^{(3|\mathcal{N})} \simeq \frac{\mathrm{ISO}(1,3|\mathcal{N})}{\mathbb{R}^3 \rtimes \left(\mathbb{R}^{0|\mathcal{N}} \rtimes (\mathrm{ISO}(2)\times\mathbb{R}\times\mathrm{SU}(\mathcal{N}))\right)}$ we find coordinates (π^A, u_+, θ_I) such that one can write

$$u_+ = u + \frac{i}{2}\theta_I \overline{\theta}_{\bar{J}} h^{I\bar{J}}$$

for a real u : chiral coordinates appear also on $\mathscr{I}^{(3|\mathcal{N})}$!

Coordinates details of the classification



Conclusion

- Compactified Minkowski can be decomposed into orbits for the Poincaré group
- $\mathscr{I}$ is one orbit, and so is an homogeneous space
- If we take advantage of the twistor representation of (super) Minkowski space, the orbit decomposition of super Minkowski works in the same way
- We find an expression for super null infinity as an homogeneous space

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- Compactified Minkowski can be decomposed into orbits for the Poincaré group
- $\mathscr{I}$ is one orbit, and so is an homogeneous space
- If we take advantage of the twistor representation of (super) Minkowski space, the orbit decomposition of super Minkowski works in the same way
- We find an expression for super null infinity as an homogeneous space
- Next step : make curved the homogeneous models, study the local geometry, super BMS group ?

Thank you for your attention !

References

[Figueroa21] José Figueroa-O'Farrill et al. "Carrollian and celestial spaces at infinity". In: *arXiv preprint arXiv:2112.03319* (2021).

[Herfray20] Yannick Herfray. "Asymptotic shear and the intrinsic conformal geometry of null-infinity". In: *Journal of Mathematical Physics* 61.7 (2020), p. 072502.

[Manin97] Yuri I Manin. *Gauge field theory and complex geometry*. Vol. 289. Springer Science & Business Media, 1997.