A revisit of the Carrollian scalar field

Second Carroll Workshop, UMONS

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Main goals

- $\Rightarrow$ Give an overview of the computation of momenta and charges in a Carrollian theory
- $\Rightarrow$ See and recall the differences with the relativistic ascendant of the theory
- $\Rightarrow$ Talk about electric/magnetic dualities
- $\Rightarrow$ Apply those results in the RT background

Highlights

1. Set up
2. Conserved charges
3. Application in Robinson-Trautman
4. Comments and outlook

Carroll structure

$\mathscr{M} = \mathrm{R} \times \mathscr{S}$ is a $d+1$-dim manifold with two fundamental quantities

$\Rightarrow$ a *degenerate metric*

$$dl^2 = a_{ij}(t, \boldsymbol{x}) dx^i dx^j.$$

$\Rightarrow$ a *field of observers* $\nu = \frac{1}{\Omega}\partial_t$.

The field of observers admits a dual form, the *clock form* $\mu = \Omega dt - \boldsymbol{b}$ with an *Ehresmann connection* $\boldsymbol{b} = b_i dx^i$.

Relativistic ascendant [Ciambelli, Marteau, Petropoulos, Petkou, Siampos 18]

Take the $c \to 0$ limit of a manifold in Randers-Papapetrou gauge

$$ds^2 = -c^2(\Omega dt^2 - b_i dx^i)^2 + a_{ij} dx^i dx^j.$$

<u>Assume:</u> all the c-dependence is explicit.

The scalar field

Our guideline

In all these considerations we want to build theories invariant under Carrollian diffeomorphisms

$$t' = t'(t, \mathrm{x}) \quad \text{and} \quad \mathrm{x}' = \mathrm{x}'(\mathrm{x}).$$

Set up

The relativistic action is

$$S = -\int_{\mathscr{M}} \mathrm{d}t\, \mathrm{d}^d x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right).$$

We extract the c-dependence assuming

$$V(\Phi) = \frac{1}{c^2} V_{\mathrm{e}}(\Phi) + V_{\mathrm{m}}(\Phi) + \mathcal{O}(c^2).$$

Electric v Magnetic theories

We are left with

$$S = \frac{1}{c^2} S_{\mathrm{e}} + S_{\mathrm{m}} + \mathcal{O}(c^2)$$

with S_{e} and S_{m} the Carrollian actions with Lagrangian densities

$$\begin{aligned} \mathcal{L}_{\mathrm{e}} &= \frac{1}{2}\left(\frac{1}{\Omega}\partial_t \Phi\right)^2 - V_{\mathrm{e}}(\Phi), \\ \mathcal{L}_{\mathrm{m}} &= -\frac{1}{2} a^{ij} \hat{\partial}_i \Phi \hat{\partial}_j \Phi - V_{\mathrm{m}}(\Phi), \end{aligned}$$

Important remark

Both Lagrangian densities are Carroll covariant thus are genuine Carrollian field theories in themselves.

The conformal scalar field [see also Baiguera, Oling Sybesma, Sogaard 22]

Potential

The field ϕ is now weight $w = \frac{d-1}{2}$ and we take a potential

$$V(\Phi) = \frac{d-1}{8d} R\phi^2.$$

This is conformal (as $T^\mu{}_\nu$). And

$$V(\Phi) = \frac{1}{c^2} V_{\mathrm{e}}(\Phi) + V_{\mathrm{m}}(\Phi) + c^2 V_{\mathrm{nd}}(\Phi)$$

with

$$\begin{aligned}
V_{\mathrm{e}}(\Phi) &= \frac{d-1}{8d}\left(\frac{2}{\Omega}\partial_t\theta + \frac{1+d}{d}\theta^2 + \xi_{ij}\xi^{ij}\right)\Phi^2, \\
V_{\mathrm{m}}(\Phi) &= \frac{d-1}{8d}\left(\hat{r} - 2\hat{\nabla}_i\varphi^i - 2\varphi^i\varphi_i\right)\Phi^2, \\
V_{\mathrm{nd}}(\Phi) &= \frac{d-1}{8d}\varpi_{ij}\varpi^{ij}\Phi^2.
\end{aligned}$$

Electric and Magnetic theories

$$S_e = \int \mathrm{d}t\, \mathrm{d}^d x \sqrt{a}\Omega \left(\frac{1}{2}\left(\frac{1}{\Omega}\hat{\mathscr{D}}_t \Phi\right)^2 - \frac{d-1}{8d}\xi_{ij}\xi^{ij}\Phi^2 \right),$$

$$S_m = \int \mathrm{d}t\, \mathrm{d}^d x \sqrt{a}\Omega \left(-\frac{1}{2}\hat{\mathscr{D}}_i \Phi \hat{\mathscr{D}}^i \Phi - \frac{d-1}{8d}\hat{\mathscr{R}}\Phi^2 \right),$$

as well as a third one $S_{nd} = -\int \mathrm{d}t\, \mathrm{d}^d x \sqrt{a}\Omega \frac{d-1}{8d}\varpi_{ij}\varpi^{ij}\Phi^2$, which has no kinetic term for Φ.

EoM

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t \frac{1}{\Omega}\hat{\mathscr{D}}_t \Phi + \frac{d-1}{4d}\xi_{ij}\xi^{ij}\Phi = 0 \quad \text{electric},$$

$$-\hat{\mathscr{D}}_i \hat{\mathscr{D}}^i \Phi + \frac{d-1}{4d}\hat{\mathscr{R}}\Phi = 0 \quad \text{magnetic},$$

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Building momenta [Ciambelli, Marteau 19 & Petkou, Petropoulos, Rivera-Betancour, Siampos 22]

From and action $\mathscr{S}$ one can define an *energy–stress tensor* Π^{ij}, an *energy flux* Π^i and an *energy density* Π, defined as:

$$\Pi^{ij} = \frac{2}{\sqrt{a}\Omega}\frac{\delta S_C}{\delta a_{ij}} \qquad \Pi^i = \frac{1}{\sqrt{a}\Omega}\frac{\delta S_C}{\delta b_i},$$

$$\Pi = -\frac{1}{\sqrt{a}}\left(\frac{\delta S_C}{\delta \Omega} + \frac{b_i}{\Omega}\frac{\delta S_C}{\delta b_i}\right)$$

satisfying the equations

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Pi + \hat{\mathscr{D}}_i\Pi^i + \Pi^{ij}\xi_{ij} = 0,$$

$$\hat{\mathscr{D}}_i\Pi^i{}_j + 2\Pi^i\varpi_{ij} + \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t\delta^i_j + \xi^i{}_j\right)P_i = 0.$$

Relevant momenta for the following

$$
\begin{aligned}
\Pi^i_{\mathrm{e}} &= 0 \\
\Pi^i_{\mathrm{m}} &= -\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi\hat{\mathscr{D}}^i\Phi + \frac{d-1}{4d}\left(\hat{\mathscr{D}}^i\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi^2 - \hat{\mathscr{D}}_j\left(\xi^{ij}\Phi^2\right)\right) \\
P^i_{\mathrm{e}} &= \Pi^i_{\mathrm{m}} \\
P^i_{\mathrm{m}} &= \Pi^i_{\mathrm{nd}} = \frac{d-1}{4d}\hat{\mathscr{D}}_j\left(\varpi^{ji}\Phi^2\right)
\end{aligned}
$$

Construction of conserved charges [Ciambelli Marteau 18] [Petkou, Petropoulos, Rivera-Betancour, Siampos 22]

The current → a scalar component κ + a Carrollian-vector set of components K^i.

The divergence takes the form

$$\mathscr{K} = \left(\frac{1}{\Omega}\partial_t + \theta\right)\kappa + \left(\hat{\nabla}_i + \varphi_i\right)K^i.$$

The charge associated with the current $(\kappa, \boldsymbol{K})$ is then

$$Q_K = \int_{\mathscr{S}} \mathrm{d}^d x \sqrt{a}\left(\kappa + b_i K^i\right),$$

we obtain the following time evolution:

$$\frac{\mathrm{d}Q_K}{\mathrm{d}t} = \int_{\mathscr{S}} \mathrm{d}^d x \sqrt{a}\Omega\mathcal{K} - \int_{\partial\mathscr{S}} \star\boldsymbol{K}\Omega,$$

where $\star\boldsymbol{K}$ is the $\mathscr{S}$-Hodge dual of $K_i \mathrm{d}x^i$.

Electric v Magnetic conservation

Take ξ a Carrollian diffeomorphism (here $\xi^i = \xi^i(\boldsymbol{x})$),

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \left(\xi^t - \xi^i \frac{b_i}{\Omega}\right) \partial_t + \xi^i \left(\partial_i + \frac{b_i}{\Omega}\partial_t\right) = \xi^{\hat{t}} \frac{1}{\Omega}\partial_t + \xi^i \hat{\partial}_i$$

This gives

$$\kappa = \xi^i P_i - \xi^{\hat{t}}\Pi, \quad K^i = \xi^j \Pi_j{}^i - \xi^{\hat{t}}\Pi^i.$$

And the divergence reads

$$\mathscr{K} = -\Pi^i \left(\left(\hat{\partial}_i - \varphi_i\right) \xi^{\hat{t}} - 2\xi^j \varpi_{ji}\right).$$

Electric

$Q_{\mathrm{e}} = \int_{\mathscr{S}} \mathrm{d}^d x \sqrt{a} \left(\kappa_{\mathrm{e}} + b_i K_{\mathrm{e}}^i \right)$

$$\kappa_{\mathrm{e}} = \xi^i \Pi_{\mathrm{m}i} - \xi^{\hat{t}} \Pi_{\mathrm{e}}, \quad K_{\mathrm{e}}^i = \xi^j \Pi_{\mathrm{e}j}^i,$$

And $\Pi_e^i = 0 \implies \mathscr{K} = 0 \implies$ all charges are conserved.

Magnetic

$Q_{\mathrm{m}} = \int_{\mathscr{S}} \mathrm{d}^d x \sqrt{a} \left(\kappa_{\mathrm{m}} + b_i K_{\mathrm{m}}^i \right)$ with

$$\kappa_{\mathrm{m}} = \xi^i \Pi_{\mathrm{nd}i} - \xi^{\hat{t}} \Pi_{\mathrm{m}}, \quad K_{\mathrm{m}}^i = \xi^j \Pi_{\mathrm{m}j}^i - \xi^{\hat{t}} \Pi_{\mathrm{m}}^i$$

Charges are conserved for configurations s.t. $\Pi_m^i = 0 \implies$ depends on the background.

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Take Robinson-Trautman in $d = 4$. The null boundary is a Carrollian manifold $\mathscr{M} = \mathbb{R} \times \mathscr{S}$, where $\mathscr{S}$ is equipped with a conformally flat $d = 2$ metric:

$$\mathrm{d}\ell^2 = \frac{2}{P^2}\mathrm{d}\zeta \mathrm{d}\bar{\zeta}$$

Carrollian data

$$\begin{aligned} \Omega &= 1 \quad b_i = 0 \quad \upsilon = \partial_t, \quad \mu = -\mathrm{d}t \quad \theta = -2\partial_t \ln P, \\ \varphi_i &= 0, \quad \varpi_{ij} = 0, \quad \xi_{ij} = 0, \quad \hat{\mathscr{R}} = 4P^2 \partial_{\bar{\zeta}} \partial_\zeta \ln P. \end{aligned}$$

Vanishing shear

If $\xi_{ij} = 0 \to a_{ij}(t, \boldsymbol{x}) = \mathscr{B}^{-2}(t, \boldsymbol{x})\tilde{a}_{ij}(\boldsymbol{x})$.

Algebras

So Carrollian algebra = Confalbegra($\tilde{a}_{ij}(\boldsymbol{x})$) $\oplus$ supertranslations.

When $\tilde{a}_{ij}(\boldsymbol{x})$ is conformally flat, recover $\mathfrak{ccar}(d+1)$ so BMS in $d = 1$ and $d = 2$.

Conformal Killing fiels [Ciambelli, Leigh, Marteau, Petropoulos 19]

The conformal Killing fields of $\mathscr{M}$ are

$$\xi_{T,Y} = (T - M_Y(C))\,\frac{1}{P}\partial_t + Y^i\partial_i,$$

where

$$C(t,\zeta,\bar{\zeta}) = \int^t \mathrm{d}\tau P(\tau,\zeta,\bar{\zeta}),$$

and M_Y is an operator acting on scalar functions $f(t,\zeta,\bar{\zeta})$ as:

$$M_Y(f) = Y^k\partial_k f - \frac{f}{2}\partial_k Y^k.$$

Remark

Recover the BMS algebra thus an infinite number of Killings and charges (not all conserved !).

Electric scalar field in RT

The electric equation of motion (1) reads as follows in the three-dimensional Carrollian spacetime under consideration:

$$\partial_t \frac{1}{P} \partial_t \frac{\Phi}{\sqrt{P}} = 0.$$

Its general solution is given in terms of two arbitrary functions $f(\zeta, \bar{\zeta})$ and $g(\zeta, \bar{\zeta})$:

$$\Phi = \sqrt{P}\,(Cf + g)\,.$$

The charges is

$$\begin{aligned} Q_{\mathrm{e}\,T,Y} &= -\mathrm{i} \int_{\mathscr{S}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \left(Y^i \left(\frac{1}{4} \partial_i (fg) - f \partial_i g \right) - \frac{T f^2}{2} \right) \\ &- \frac{1}{4} \int_{\partial \mathscr{S}} \star \boldsymbol{Y} C f^2 P^2. \end{aligned}$$

Magnetic scalar field in RT

The magnetic equation (1) is

$$4\partial_\zeta \partial_{\bar\zeta} \Phi = \Phi \partial_\zeta \partial_{\bar\zeta} \ln P.$$

Conservation $\rightarrow$ two cases to consider

$\Rightarrow$ $\Pi^i_m = 0$

$\Rightarrow$ $\left(\hat\partial_i - \varphi_i\right) \xi^{\hat t} - 2\xi^j \varpi_{ji} = 0$

Vanishing energy flux

Conformally stationary scalars of the form $\Phi = \sqrt{P} g(\zeta, \bar\zeta)$, where $g(\zeta, \bar\zeta)$ is further determined by solving the magnetic equation of motion.

Magnetic charges non-zero and conserved for all ξ.

Vanishing of extra condition

This implies

$$T = SP + M_Y(C),$$

where $S = S(t) \rightarrow$ huge restriction on the allowed ξ.
Only 1 charge conserved

$$Q_{\mathrm{m}\,S} = -S \int_{\mathscr{S}} \frac{\mathrm{d}\zeta \mathrm{d}\bar{\zeta}}{P^2} \Pi_{\mathrm{m}}.$$

$\rightarrow$ total energy, but $\Pi_{\mathrm{m}} = \frac{1}{2d} \hat{\mathscr{D}}_i \left(\Phi \hat{\mathscr{D}}^i \Phi \right)$ on shell so this only charge vanishes.

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Some conclusions

Why such a spliting ?

Carrollian (conformal) isometry $\rightarrow$ invariance of $a_{ij}(\boldsymbol{x}, t)$ and $= \frac{1}{\Omega}\partial_t$, but not that of $\mu = \Omega dt - \boldsymbol{b}$.
Time (supported by ν) and space (associated with μ) directions behave differently and this ultimately reveals in the conservation properties of electric versus magnetic dynamics.

On the charges

Not all Carrollian (conformal) Killing vectors give rise to a conserved quantity. Much more isometries than conserved quantities !

End

Thanks for your attention !