Conformal Carrollian geometry at null infinity

Xavier BEKAERT

Institut Denis Poisson (Tours)

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Carrollian geometry: a brief history

- **1965:** J.-M. Lévy-Leblond (and, independently, S. Gupta) investigated the Inönu-Wigner contraction of the Poincaré group that arises in the ultrarelativistic limit ($c \to 0$), which Lévy-Leblond dubbed "Carroll group" as a tribute to the exotic (mad?) causal features of spacetime in this limit.

Carrollian geometry: a brief history

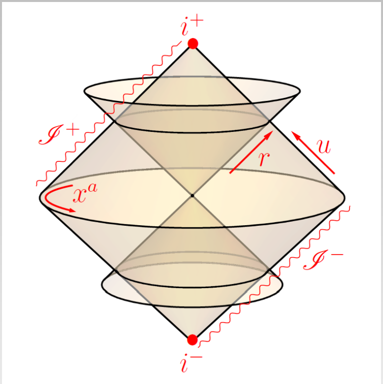
- **1965:** J.-M. Lévy-Leblond (and, independently, S. Gupta) investigated the Inönu-Wigner contraction of the Poincaré group that arises in the ultrarelativistic limit ($c \to 0$). He called it "Carroll group" as a tribute to the exotic (mad?) features of this limit.
- **1965-1977:** Penrose and Geroch introduced the intrisinc boundary approach to the celebrated Bondi-Mezner-Sachs (BMS) group of asymptotic symmetries for asymptotically flat spacetimes.
- **1979:** Henneaux investigated (from a Hamiltonian perspective) the ultrarelativistic limit of spacetime geometry and dynamical gravity.
- **2014:** Duval, Gibbons, Horvathy revisited the Geroch-Penrose definition of BMS group and identified the latter with the natural conformal extension of the Carroll group, and reinvented the Henneaux definition of what they decided to call a "Carrollian manifold," i.e. a manifold endowed with a degenerate metric of null signature (0,+,...,+) whose radical defines the fundamental vector field of a principal line bundle.

Outline

1. Introduction
   - Carrollian geometry: a brief history
   - Outline

2. Conformal Carrollian geometry
   - Principal bundle geometry
   - Carrollian geometry
   - Conformal Carrollian geometry
   - Generalised BMS geometry

Intrinsic and geometric view of BMS symmetries



Intrinsic view

Although BMS group is often discussed from the point of view of asymptotic symmetries of a bulk spacetime, it can be formulated in an

- intrinsic (*i.e.* purely from the boundary) and
- geometric (*i.e.* global and coordinate-free) way.

This point of view on BMS group

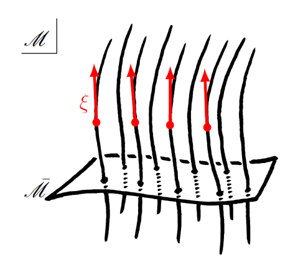
- goes back to Penrose (1965) and Geroch (1977)
- allows to interpret the BMS group as a conformal extension of Carroll group (Duval-Gibbons-Horvathy, 2014)

Principal bundle (aka ambient) geometry

Fundamental vector field

Fundamental vector field: (essentially) equivalent data

- Nowhere vanishing vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on a manifold $\mathscr{M}$
- Congruence of parametrised curves from $\mathbb{R}$ to $\mathscr{M}$
- Principal $\mathbb{R}$-bundle $\mathscr{M}$ with fundamental vector field ξ



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- The curves are the integral lines of the fundamental vector field; they are also the orbits of the $\mathbb{R}$-action on $\mathscr{M}$.
- The space $\bar{\mathscr{M}}$ of such orbits is the base manifold of the principal bundle

$$\bar{\mathscr{M}} = \mathscr{M} / \mathbb{R}$$

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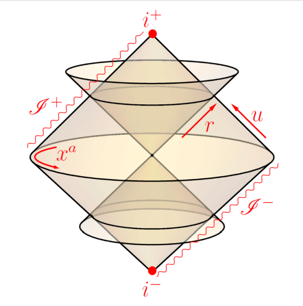
Local expression: there exist a coordinate system (u, x^a) such that

- Fundamental vector field $\xi = \frac{\partial}{\partial u}$
- Curves $x^a = x^a_0$ parametrised by u
- $\mathbb{R}$-action $u \to u - u_0$ $(u_0 \in \mathbb{R})$
- Fibration $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}} : (u, x^a) \mapsto x^a$

Fundamental vector field

Example 1 : Future null infinity $\mathscr{I}^+_{d+1}$ at the conformal boundary of the compactification of Minkowski spacetime $\mathbb{R}^{d+1,1}$

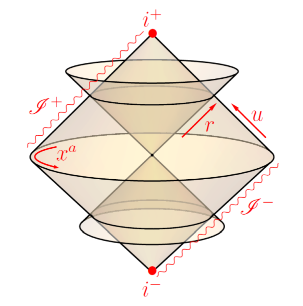
- Coordinates (u, x^a) on $\mathscr{I}^+_{d+1} \cong \mathbb{R} \times S^d$
- Fundamental vector field $\xi = \frac{\partial}{\partial u}$ is null
- Null rays generating the cone
- $\mathbb{R}$-action $u \to u - u_0$ $(u_0 \in \mathbb{R})$
- Fibration $\pi : \mathscr{I}^+_{d+1} \twoheadrightarrow S^d : (u, x^a) \mapsto x^a$



Fundamental vector field

Example 2: Möbius model (projective null cone)

- Inversion $x^\mu = \frac{x^\mu}{x^2} \quad \Rightarrow \quad \mathscr{I}^\pm \leftrightarrow \mathscr{N}^\mp$
- Past lightcone $\mathscr{N}^- \subset \mathbb{R}^{d+1,1}$ of the origin of Minkowski spacetime
- Coordinates (u, x^a) on $\mathscr{N}^- \cong \mathbb{R} \times S^d$
- Etc (idem as $\mathscr{I}^+$)



Invariant lift of a function

Consider a principal $\mathbb{R}$-bundle $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}}$ with fundamental vector field ξ.

$$f = \pi^* \bar{f} = \bar{f} \circ \pi \in C^\infty(\mathscr{M}) \,, \qquad \mathcal{L}_\xi f = 0 \,,$$

which leads to the bijection

$$C^\infty_{inv}(\mathscr{M}) \cong C^\infty(\bar{\mathscr{M}}) \,.$$

Projection on the base manifold

Consider a principal $\mathbb{R}$-bundle $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}}$ with fundamental vector field ξ.

- **Projectable vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = f\,\xi$ where $f \in C^\infty(\mathscr{M})$
- **Super-projectable vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = f\,\xi$ with $\mathcal{L}_\xi f = 0$
- **Invariant vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = 0$

Remark: *Vertical* vector fields, i.e. $X = h\,\xi$ with $h \in C^\infty(\mathscr{M})$, are necessarily projectable.

Projection on the base manifold

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- **Invariant vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = 0$

Remark: *Projectable* vector fields are infinitesimal automorphisms of the fibre bundle

$$u' = u + \epsilon\, F(u,x)\,, \quad x' = x + \epsilon\, G(x)\,.$$

Projection on the base manifold

Consider a principal $\mathbb{R}$-bundle $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}}$ with fundamental vector field ξ.

- **Projectable vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = f\,\xi$ where $f \in C^\infty(\mathscr{M})$
- **Super-projectable vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = f\,\xi$ with $\mathcal{L}_\xi f = 0$
- **Invariant vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $\mathcal{L}_\xi X = 0$

Remark: *Invariant* vector fields are infinitesimal automorphisms of the principal $\mathbb{R}$-bundle,

$$u' = u + \epsilon\, F(x)\,, \quad x' = x + \epsilon\, G(x)\,.$$

Projection on the base manifold

Example: Invariant vertical vector fields ($X = h\,\xi$ with $\mathcal{L}_\xi h = 0$) generate vertical automorphisms of the principal $\mathbb{R}$-bundle

$$u' = u + f(x)\,, \quad x' = x\,,$$

which are interpreted as "supertranslations" in the BMS context.

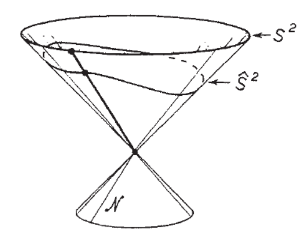


Fig. 2 in R. Penrose, 1974

Pullback from the base manifold

Consider a principal $\mathbb{R}$-bundle $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}}$
with fundamental vector field ξ.

- **Invariant differential one-form:** $A \in \Omega^1(\mathscr{M})$ such that $\mathcal{L}_\xi A = 0$
- **Horizontal differential one-form:** $A \in \Omega^1(\mathscr{M})$ such that $A \cdot \xi = 0$
- **Basic differential one-form:** invariant & horizontal
  $\Leftrightarrow A = \pi^* \bar{A}$ with $\bar{A} \in \Omega^1(\bar{\mathscr{M}})$

These definitions generalise to covariant tensor fields (e.g. the Carrollian metric).

Pullback from the base manifold

Consider a principal $\mathbb{R}$-bundle $\pi : \mathscr{M} \twoheadrightarrow \bar{\mathscr{M}}$ with fundamental vector field ξ.

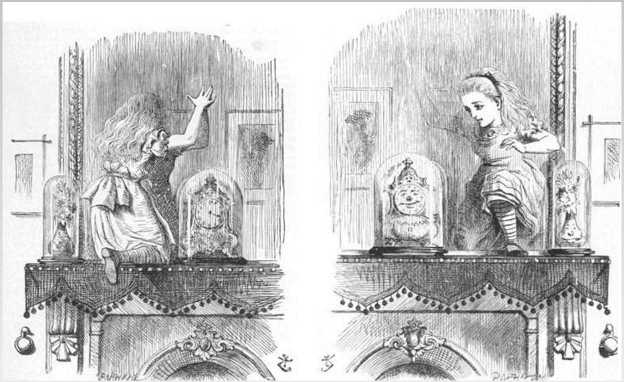
- **Ehresmann connection one-form:** invariant differential one-form $A \in \Omega^1_{\text{inv}}(\mathscr{M})$ such that $A \cdot \xi = 1$
- **Horizontal vector field:** $X \in \mathfrak{X}(\mathscr{M})$ such that $A \cdot X = 0$

A flat Ehresmann connection defines a foliation of $\mathscr{M}$ by horizontal leaves $\cong \bar{\mathscr{M}}$.

Carrollian geometry

Carrollian structure

Original motivation: ultrarelativistic (aka Carrollian) structures are the duals of nonrelativistic (aka Galilean) structures [Duval, Gibbons, Horvathy, 2014]



Carrollian structure :

Field of observers

&

Carrollian metric

Timelike metric structure

Field of observers: fundamental vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on the spacetime manifold $\mathscr{M}$ fibred over the absolute space $\bar{\mathscr{M}}$.

Provides a distinction between the type of vectors in Carroll geometry:

$$\begin{cases} V^\mu = f\,\xi^\mu \quad \text{with} & \begin{cases} f \neq 0 & \textbf{Timelike (or Vertical)} \\ f > 0 & \textbf{Future-oriented} \end{cases} \\ V^\mu \neq f\,\xi^\mu & \quad\quad \textbf{Spacelike} \end{cases}$$

Timelike metric structure

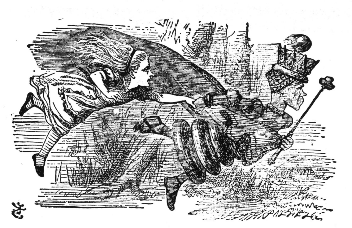
Field of observers: fundamental vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on the spacetime manifold $\bar{\mathscr{M}}$ fibred over the absolute space $\mathscr{M}$.

$\Rightarrow$ The integral lines of the fundamental vector field are the only admissible worldlines and they are vertical: all inertial observers are at rest.

Timelike metric structure

Field of observers: fundamental vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on the spacetime manifold $\mathscr{M}$ fibred over the absolute space $\bar{\mathscr{M}}$.

$\Rightarrow$ The integral lines of the fundamental vector field are the only admissible worldlines and they are vertical: all observers are at rest. This property is another motivation for the nickname "Carroll" (Dyson, 1965), cf the Red Queen: "it takes all the running you can do, to keep in the same place."



Timelike metric structure

Field of observers: fundamental vector field $\xi = \xi^\mu \partial_\mu \neq 0$ on the spacetime manifold $\mathscr{M}$ fibred over the absolute space $\bar{\mathscr{M}}$.

An affine parameter u of this congruence of Carroll worldlines (i.e. $\xi = \partial/\partial u$) is a Carroll time.

Spacelike metric structure

Carrollian metric: Positive semi-definite metric γ on the spacetime $\mathscr{M}$ whose kernel is spanned by the fundamental vector field

$$\begin{cases} \gamma_{\mu\nu} V^\mu W^\nu \geqslant 0 \\ \gamma_{\mu\nu} V^\mu = 0 \quad \Leftrightarrow \quad V^\mu = f\, \xi^\mu \end{cases}$$

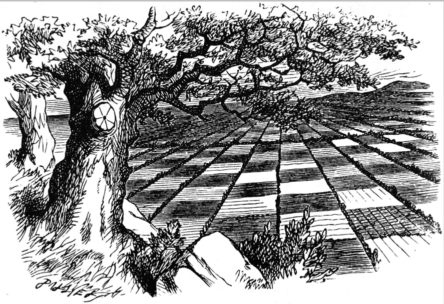
Remark: There is a one-to-one correspondence between

- *invariant* Carrollian metrics $\gamma_{\mu\nu}$ on $\mathscr{M}$ and
- Riemannian metrics $\bar{\gamma}_{ab}$ on the base $\bar{\mathscr{M}}$

since an invariant Carrollian metric is basic, $\gamma = \pi^* \bar{\gamma}$.

Spacelike metric structure

An invariant Carrollian metric allows to measure distances and angles on the base manifold $\bar{\mathscr{M}}$.



Carrollian structure

Definition (Henneaux, 1979)

(Invariant) Carrollian structure: two data

1. Field of observers
2. (Invariant) Carrollian metric

One will focus on *invariant* Carrollian structures, so this assumption will sometimes be implicitly assumed from now on.

Example 1: Flat Carrollian spacetime

- Cartesian coordinates (u, x^i) on $\mathbb{R} \times \mathbb{R}^d$
- Time translation generator $\xi = \frac{\partial}{\partial u}$
- Flat Carrollian metric = pullback of the metric on the Euclidean space

$$ds^2_{\mathbb{R}\times\mathbb{R}^d} = \delta_{ij}\, dx^i dx^j = d\ell^2_{\mathbb{R}^d}$$

Carrollian structure

Example : Future null infinity in Bondi frame

- Coordinates (u, x^a) on $\mathscr{I}^+_{d+1} \cong \mathbb{R} \times S^d$
- Null vector field $\xi = \frac{\partial}{\partial u}$
- Carrollian metric = pullback of the metric on the unit sphere

$$ds^2_{\mathscr{I}^+_{d+1}} = \gamma_{ab}(x)\, dx^a dx^b = d\ell^2_{S^d}$$

Carrollian structure

Carrollian structure: following two data

1. Field of observers
2. Carrollian metric

Remarks:

- Intrisic: A Carrollian structure has exactly the same number of independent components as a Lorentzian structure (i.e. a relativistic metric) in the same dimension.
- Extrinsic: Any Carrollian spacetime can be embedded inside a Lorentzian spacetime of one extra dimension.

Carrollian structure

The pullback of a Lorentzian structure on a null hypersurface in a relativistic spacetime defines a Carrollian structure on this submanifold.

Example 1: Null hyperplane $v = v_0$ in Minkowski spacetime $\mathbb{R}^{d+1,1}$ (pullback of metric $ds^2 = 2\,du dv + \delta_{ij} dx^i dx^j \Rightarrow$ flat Carrollian spacetime $\xi = \frac{\partial}{\partial u}$ and $\gamma_{ij} = \delta_{ij}$)

Example 2: Future/Past lightcone $\mathscr{N}^\pm$ in Minkowski spacetime

Example 3: Future/Past null infinity $\mathscr{I}^\pm$ at the conformal boundary of compactified Minkowski spacetime

Carrollian isometries

Carrollian isometry: diffeomorphism of $\mathscr{M}$ preserving the

1. Field of observers $\xi' = \xi$
2. Carrollian metric $\gamma' = \gamma$

Remark: For an invariant Carrollian structure, these Carrollian isometries project onto isometries of the Riemannian metric on the base.

Carrollian isometries

Carrollian isometry: diffeomorphism of $\mathscr{M}$ preserving the

1. Field of observers $\xi' = \xi$
2. Carrollian metric $\gamma' = \gamma$

Example: Vertical automorphisms of the principal $\mathbb{R}$-bundle

$$u' = u + f(x)\,, \quad x' = x\,,$$

which are interpreted as “supertranslations” in the BMS context.

In particular, *Carrollian time translations* and *Carrollian boosts* at null infinity arise from bulk (time and, respectively, spatial) translations in the interior Minkowski spacetime.

Carrollian isometries

Carrollian isometry: diffeomorphism of $\mathscr{M}$ preserving the

1. Field of observers $\xi' = \xi$
2. Carrollian metric $\gamma' = \gamma$

Remark: The algebra of Carrollian isometry generators has a structure of semi-direct sum

$$\mathfrak{carr\,isom}(\mathscr{M}) \cong \mathfrak{isom}(\bar{\mathscr{M}}) \in C^\infty(\bar{\mathscr{M}})$$

Strong Carrollian structure

Definition

Strong Carrollian structure: three data

1. Field of observers
2. Carrollian metric
3. Compatible affine connection

Example: Flat Carrollian spacetime

- Cartesian coordinates (u, x^i) on $\mathbb{R} \times \mathbb{R}^d$
- Time translation generator $\xi = \frac{\partial}{\partial u}$
- Flat Carrollian metric = pullback of the metric on Euclidean space

$$ds^2_{\mathbb{R}\times\mathbb{R}^d} = \delta_{ij}\, dx^i dx^j = d\ell^2_{\mathbb{R}^d}$$

- Flat affine connection $\Gamma^\rho_{\mu\nu} = 0$

Strong Carrollian isometries

Strong Carrollian isometry: diffeomorphism of $\mathscr{M}$ preserving the

1. Field of observers $\xi' = \xi$
2. Carrollian metric $\gamma' = \gamma$
3. Affine connection $\nabla' = \nabla$

Remark 1: The Lie algebra of strong Carrollian isometry generators of flat Carrollian spacetime is the finite-dimensional Carroll algebra, i.e. the Inönu-Wigner contraction of the Poincaré group arising in the ultrarelativistic limit

$$\mathfrak{iso}(d,1) \xrightarrow{c\to 0} \mathfrak{carr}(d,1)$$

In fact, preserving the flat connection only leaves affine transformations, thereby leaving only the Carrollian time translations and Carrollian boosts

$$u' = u + a + b_i\, x^i\,, \quad x' = x\,,$$

among the vertical automorphisms.

Strong Carrollian isometries

Strong Carrollian isometry: diffeomorphism of $\mathscr{M}$ preserving the

1. Field of observers $\xi' = \xi$
2. Carrollian metric $\gamma' = \gamma$
3. Affine connection $\nabla' = \nabla$

Remark 2: The Carroll algebra, has a structure of semidirect sum

$$\mathfrak{iso}(d,1) = \mathfrak{so}(d,1) \inplus \mathbb{R}^{d+1} \quad \xrightarrow{c\to 0} \quad \mathfrak{carr}(d,1) = (\mathfrak{iso}(d) \inplus \mathbb{R}^d) \oplus \mathbb{R}$$

wich can be understood as follows: (i) the time translation generator ∂_u is central in the Carroll algebra and (ii) the homogeneous Carroll algebra has itself a structure of semidirect sum

$$\mathfrak{so}(d,1) \xrightarrow{c\to 0} \mathfrak{iso}(d)$$

since the Carroll boost generators $\hat{B}_i = x_i \partial_u$ commute with each other and transform as vectors under rotations.

Aristotelian structure

Definition

Aristotelian structure: three data

1. Field of observers
2. (Invariant) Carrollian metric
3. (Principal) Ehresmann connection

Example: Flat Aristotelian spacetime

- Cartesian coordinates (u, x^i) on $\mathbb{R} \times \mathbb{R}^d$
- Time translation generator $\xi = \frac{\partial}{\partial u}$
- Flat Carrollian metric = pullback of the metric on Euclidean space

$$ds^2_{\mathbb{R}\times\mathbb{R}^d} = \delta_{ij}\, dx^i dx^j = d\ell^2_{\mathbb{R}^d}$$

- Flat Ehresmann connection $A = du$

Aristotelian isometries

Definition (Penrose, 1968)

Aristotelian isometries: diffeomorphism of $\mathscr{M}$ preserving the

1. Field of observers
2. Carrollian metric
3. Ehresmann connection

Example: The Lie algebra of isometries of the flat Aristotelian spacetime $\mathbb{R} \oplus \mathfrak{iso}(d)$ is the "static" (i.e. without boosts) algebra in the classification by Bacry & Lévy-Leblond (1968) of kinematical algebras.

Bondi-Metzner-Sachs as Conformal Carroll

Brief interlude on densities and weights

Definition

(Volumic) density with weight w: scalar field $\varphi(x)$ with transformation law

$$\phi'(x') = \left| \det\left(\frac{\partial x'^a}{\partial x^b}\right) \right|^{-w} \phi(x) .$$

More generally, a tensor-valued (volumic) density of weight w is a tensor field whose usual transformation law under reparametrisations involves an extra Jacobian factor to the power w.

The corresponding infinitesimal transformation law is

$$\delta\phi = \mathcal{L}_X \phi + w\,\partial_a X^a\,\phi\,,$$

where $\mathcal{L}_X$ is the Lie derivative along X acting on the tensor field ϕ; for a scalar field ($w = 0$) it reduces to $\delta\phi = X^a \partial_a \phi$.

Brief interlude on densities and weights

Definition

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$$\phi'(x') = \left| \det \left(\frac{\partial x'^a}{\partial x^b} \right) \right|^{-w} \phi(x) \,.$$

More generally, a tensor-valued (volumic) density of weight w is a tensor field whose usual transformation law under reparametrisations involves an extra Jacobian factor to the power w.

Scalar (volumic) densities of weight $w = 1$ on a manifold (not necessarily orientable) are the objects that can be integrated in a coordinate-independent way.

Brief interlude on densities and weights

Independently of a density's behaviour under diffeomorphisms, one can also define a notion of weight under Weyl transformations.

Definition

Conformal densities: The transformation law of a (scalar or tensor) *conformal density* ψ of conformal weight ω under Weyl transformations is given by

$$g_{ab}(x) \to g'_{ab}(x) = \Omega^2(x)\, g_{ab}(x)\,, \qquad \psi'(x) \;=\; \Omega(x)^{\omega}\, \psi(x)\,.$$

Note that a field may well be a volumic density and a conformal density simultaneously.
For instance, the metric g_{ab} is a tensor density of volumic weight zero and conformal weight two.
Similarly, the volume density $\sqrt{g}$ on a manifold of dimension d is a scalar volumic density with volumic weight $w = 1$ and conformal weight $\omega = d$.

Conformal Carrollian structure

Definition (Penrose, 1965; Geroch, 1977)

Conformal Carrollian structure: equivalence class $[\xi, \gamma]$ of Carrollian structures, i.e. pairs (ξ, γ), with respect to the equivalence relation

1. Field of observers $\xi \sim \Omega^{-1}\xi$
2. (Invariant) Carrollian metrics $\gamma \sim \Omega^2\gamma$ (with $\mathcal{L}_\xi\Omega = 0$)

where $\Omega > 0$.

Remark: In the invariant case, the conformal Carrollian metric $[\gamma]$ on the Carrollian spacetime $\mathscr{M}$ is the pullback of the conformal metric $[\bar{\gamma}]$ on the base $\bar{\mathscr{M}}$.

Conformal Carrollian isometries

Conformal Carrollian isometry: diffeomorphism of $\mathscr{M}$ such that

1. (Conformal rescaling) $\xi' = \Omega^{-1}\xi$
2. (Conformal isometry) $\gamma' = \Omega^2\gamma$

with $\mathcal{L}_\xi \Omega = 0$.

Conformal Carrollian isometries

Conformal Carrollian isometry: diffeomorphism of $\mathscr{M}$ such that

1. (Conformal rescaling) $\xi' = \Omega^{-1}\xi$
2. (Conformal isometry) $\gamma' = \Omega^2\gamma$

with $\mathcal{L}_\xi\Omega = 0$.

Remark: For an invariant conformal Carrollian structure $[\xi, \gamma]$, these conformal Carrollian isometries project onto conformal isometries of the conformal metric $[\bar{\gamma}]$ on the base $\bar{\mathscr{M}}$.

Conformal Carrollian isometries

Conformal Carrollian isometry: diffeomorphism of $\mathscr{M}$ such that

1. (Conformal rescaling) $\xi' = \Omega^{-1}\xi$
2. (Conformal isometry) $\gamma' = \Omega^2\gamma$

with $\mathcal{L}_\xi \Omega = 0$.

Example: For null infinity $\mathscr{I}$

Theorem ((Penrose, 1965) revisited (Duval-Gibbons-Horvathy, 2014))

BMS transformations = Conformal Carrollian isometries

Conformal Carrollian isometries

The group of conformal Carrollian isometries of null infinity coincides with the BMS group

$$BMS_{d+2} = SO(d+1,1) \ltimes C^\infty(S^d)$$

where the elements of $C^\infty(S^d)$ transform as densities of conformal weight -1 under the elements of the Lorentz group $SO(d+1,1)$ realised as global conformal transformations of the celestial sphere S^d.
The conformal Carrollian isometries of null infinity $\mathscr{I}_{d+1} = \mathbb{R} \times S^d$ project onto conformal isometries of the celestial sphere S^d. In fact, there is a canonical surjective morphism of groups:

$$BMS_{d+2} \twoheadrightarrow SO(d+1,1)$$

whose kernel is the normal subgroup of vertical automorphisms of the principal $\mathbb{R}$-bundle. In other words, there is a canonical injective morphism of groups:

$$C^\infty(S^d) \hookrightarrow BMS_{d+2}$$

Conformal Carroll-Killing vector field

Conformal Carroll-Killing vector field: $X \in \mathfrak{X}(\mathscr{M})$ such that

1. (super-projectable) $\mathcal{L}_X \xi = f\, \xi$ with $\mathcal{L}_\xi f = 0$
2. (conformal Killing) $\mathcal{L}_X \gamma = -2f\, \gamma$

Conformal Carroll-Killing vector field

Consider an invariant conformal Carrollian structure.

The projection $\bar{X} = \pi_*(X)$ on the base $\bar{\mathscr{M}}$ of a conformal Carroll-Killing vector field X on $\mathscr{M}$ is a conformal Killing vector field $\bar{X}$ on $\bar{\mathscr{M}}$.

Conformal Carroll-Killing vector field: $X \in \mathfrak{X}(\mathscr{M})$ such that

1. (super-projectable) $\mathcal{L}_X \xi = f\,\xi$ with $\mathcal{L}_\xi f = 0$
2. (conformal Killing) $\mathcal{L}_{\bar{X}} \bar{\gamma} = -2\bar{f}\bar{\gamma}$

Conformal Carroll-Killing vector field

The conformal Carroll-Killing vector fields on $\mathscr{I}_{d+1} \cong \mathbb{R} \times S^d$ span the (extended) BMS algebra

$$(\mathfrak{e})\mathfrak{bms}_{d+2} = \mathfrak{conf}(S^d) \ni C^\infty(S^d)$$

where the elements of $C^\infty(S^d)$ transform as conformal densities of weight -1 under

$$\mathfrak{conf}(S^d) \cong \begin{cases} \mathfrak{so}(d+1,1) & \text{for} \quad d \geqslant 3, \\ \mathfrak{X}(S^1) \oplus \mathfrak{X}(S^1) & \text{for} \quad d = 2, \\ \mathfrak{X}(S^1) & \text{for} \quad d = 1. \end{cases}$$

Generalised BMS geometry

Campiglia-Laddha structure

Let $(\mathscr{M}, \xi)$ be a principal $\mathbb{R}$-bundle and assume $\mathscr{M}$ is orientable. Then an **invariant volume form** is a nowhere-vanishing top-form $\varepsilon \in \Omega^{d+1}(\mathscr{M})$ such that $\mathcal{L}_\xi \epsilon = 0$.

Campiglia-Laddha structure: equivalence class $[\xi, \varepsilon]$ of pairs (ξ, ε) with respect to the equivalence relation

1. Field of observers $\xi \sim \Omega^{-1}\xi$
2. (Invariant) volume forms $\varepsilon \sim \Omega^{d+1}\varepsilon$ (with $\mathcal{L}_\xi \Omega = 0$)

Generalised BMS transformations

Generalised conformal maps: diffeomorphism of $\mathscr{M}$ such that

1. $\xi' = \Omega^{-1}\xi$
2. $\varepsilon' = \Omega^{d+1}\varepsilon$

with $\mathcal{L}_\xi \Omega = 0$.

Example: The generalised conformal maps on $\mathscr{I}_{d+1} \cong \mathbb{R} \times S^d$ span the generalised BMS algebra

$$\mathfrak{gbms}_{d+2} = \mathfrak{X}(S^d) \Subset C^\infty(S^d)$$

where the elements of $C^\infty(S^d)$ transform as volumic densities of weight $-1/d$.

Generalised BMS transformations

This leads to the hierarchy

$$\mathfrak{iso}(d+1,1) \subset \mathfrak{bms}_{d+2} \subseteq \mathfrak{ebms}_{d+2} \subseteq \mathfrak{gbms}_{d+2} \subset \mathfrak{X}_{\mathsf{spro}}(\mathscr{I}_{d+1}) \subset \mathfrak{X}_{\mathsf{pro}}(\mathscr{I}_{d+1})$$

Conclusion

Summary

(main take away)

At null infinity, intrinsic & geometric perspective

BMS transformations = Conformal Carrollian isometries

Thank you for your attention

All illustrations of Alice are from
John Tenniel (1820-1914)