Continuous media on Newton-Cartan spacetime

§1. Basic motivations

Basic equation $\nabla_\mu T^{\mu\nu} = 0$ for relativistic fluids with

$$T_{\mu\nu} = \varepsilon \frac{u_\mu u_\nu}{c^2} + p\left(\frac{u_\mu u_\nu}{c^2} + g_{\mu\nu}\right) + \tau_{\mu\nu} + \frac{u_\mu q_\nu + u_\nu q_\mu}{c^2}$$

Two approaches

* limit $c \to \infty$ on $\nabla_\mu T^{\mu\nu} = 0$ — physical & concrete
* work directly on a "Galilean spacetime" — Newton-Cartan — abstract & general
  - geometric variable
  - conjugate momenta

phenomenological equation $\longleftrightarrow$ invariance under diffeos

Reminder on the relativistic side

$$\delta S = \int \frac{d^{d+1}x}{c} \frac{\sqrt{-g}}{2} T^{\mu\nu} \delta g_{\mu\nu} \qquad T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

$$-\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$$= -\int \frac{d^{d+1}x}{c} \sqrt{-g}\, T^{\mu\nu} \nabla_\mu \xi_\nu = -\int \frac{d^{d+1}x}{c} \sqrt{-g}\, \nabla_\mu (T^{\mu\nu} \xi_\nu) + \int \frac{d^{d+1}x}{c} \sqrt{-g}\, \xi_\nu \nabla_\mu T^{\mu\nu}$$

$$= \underbrace{-\int \frac{d^{d+1}x}{c} \partial_\mu \left(\sqrt{-g}\, T^{\mu\nu} \xi_\nu\right)}_{\text{bdy term}} + \int \frac{d^{d+1}x}{c} \sqrt{-g}\, \xi_\nu \nabla_\mu T^{\mu\nu} \qquad \forall \xi_\nu$$

$$\Rightarrow \nabla_\mu T^{\mu\nu} = 0$$

Main assumption behind this computation: Levi-Civita connection

otherwise
* $\mathcal{L}_\xi g_{\mu\nu} \neq \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$
* more degrees of freedom $\Rightarrow$ hypermomentum

* For a metric compatible connection with torsion

$$\mathcal{L}_{\xi}\, g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - 2\, T_{(\mu\nu)\rho}\, \xi^\rho$$

$$T_{\mu\nu\rho} = g_{\mu\lambda}\, T^\lambda{}_{\nu\rho} \qquad T^\lambda{}_{\nu\rho} = 2\, \Gamma^\lambda{}_{[\nu\rho]} \text{ torsion}$$

* $$\delta S = \int \frac{d^{d+1}x}{c} \sqrt{g} \left\{ \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + \Theta_\rho{}^{\mu\nu}\, \delta \Gamma^\rho{}_{\mu\nu} \right\}$$

$$\Theta_\rho{}^{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta \Gamma^\rho{}_{\mu\nu}} \quad \text{hypermomentum.}$$

See Petkou, Siampos... Although this might look unnatural in relativistic physics where we like Levi-Civita, it might be unavoidable in more exotic situations such as Galilean or Carrollian systems

§2 Newton–Cartan spacetime

* The $c \to \infty$ paradigm — the limit depends on the geometry!

$$ds^2 = \eta_{AB}\, \theta^A \theta^B = -c^2 \Omega^2 dt^2 + a_{ij}(dx^i - w^i dt)(dx^j - w^j dt)$$

$$\partial_s^2 = \eta^{AB} e_A e_B = \frac{-1}{c^2 \Omega^2} (\partial_t + w^i \partial_i) + a^{ij} \partial_i \partial_j$$

Assuming no dependence on c except for the visible one

$$\partial_s^2 \text{ (the cometric)} \xrightarrow[c \to \infty]{} a^{ij} \partial_i \partial_j = \partial_e^2$$

is degenerate with kernel $\tau = \Omega\, dt$

and dual vector

$$v = \frac{1}{\Omega} (\partial_t + w^i \partial_i)$$

* A Newton-Cartan spacetime is a manifold equipped with a degenerate cometric. The kernel of this cometric is a clock form τ. The dual vector v brings a further piece of information into the game, called field of observers (further because it is not included in the metric).

Our philosophy: split time and space and use a non-degenerate metric on the spatial section

- Pure Cartan formalism:

$$\partial \ell^2 = \delta^{ab}\, \hat{e}_a\, \hat{e}_b \qquad \tau = -\mu \;,\; v$$

$$d\ell^2 = \delta_{ab}\, \hat{\mu}^a\, \hat{\mu}^b \qquad \langle \tau, v\rangle = 1$$

Tensors carry spatial indices $\{a\}$, which can be lowered and raised with δ^{ab}, δ_{ab}

$$\langle \tau, \hat{e}_a\rangle = 0$$

$$\langle \hat{\mu}^a, v\rangle = 0$$

$\Rightarrow$ this is a reduction wrt $SO(d)$

$$\langle \hat{\mu}^a, e_b\rangle = \delta^a{}_b$$

- Hybrid formalism

$$\partial \ell^2 = a^{ij}\, \partial_i\, \partial_j \qquad \tau = \Omega\, dt$$

$$d\ell^2 = a_{ij}\, \hat{dx}^i\, \hat{dx}^j \qquad v = \frac{1}{\Omega}\hat{\partial}_t = \frac{1}{\Omega}(\partial_t + w^i\partial_i)$$

Tensors carry spatial indices $\{i\}$, which can be lowered and raised with a_{ij} and a^{ij}

$$\hat{dx}^i = dx^i - w^i dt$$

$\Rightarrow$ this is a reduction wrt the Galilean diffeomorphisms

$$\begin{cases} t \to t' = t'(t) \\ x^i \to x'^i = x'^i(t, \vec{x}) \end{cases}$$

$$\langle \tau, \partial_i\rangle = 0$$

$$\langle \hat{dx}^i, v\rangle = 0$$

$$\langle \hat{dx}^i, \partial_j\rangle = \delta^i{}_j$$

$$J = \frac{\partial t'}{\partial t} \qquad j^i = \frac{\partial x'^i}{\partial t} \qquad J^i{}_j = \frac{\partial x'^i}{\partial x^j}$$

$$\Omega' = \frac{\Omega}{J} \qquad a'_{ij} = a_{kl}\, J^{-1\,k}{}_i\, J^{-1\,l}{}_j$$

$$w'^k = \frac{1}{J}\left(J^k{}_i w^i + j^k\right)$$

τ and v are invariant whereas

$$\partial'_i = J^{-1j}{}_i\, \partial_j \qquad \hat{d}x'^i = J^i{}_j\, \hat{d}x^j$$

(time direction are scalars basically)

We abandon in this formalism general covariance, which is nevertheless non-natural.

Galilean group: action in the tangent space

Action on the orthonormal basis of the Lorentz group $\{e_A\}$:

$$e_0 = \frac{1}{c} v \qquad \theta^0 = c\,\tau \qquad \partial_s^2 = \eta^{AB} e_A e_B$$

$$e_a = e_a{}^i \partial_i \qquad \delta^{ab} e_a{}^i e_b{}^j = a^{ij} \quad \text{etc.}$$

$$e'_A = \Lambda^B{}_A\, e_B$$

* rotations $\Lambda^A{}_B = \begin{pmatrix} 1 & 0 \\ 0 & r^a{}_b \end{pmatrix}$

* boosts $\Lambda^A{}_B = \begin{pmatrix} \Gamma & \Gamma \frac{A_b}{c} \\ \Gamma \frac{A^a}{c} & \delta^a{}_b + \frac{A^a A_b}{\vec{A}^2}(\Gamma - 1) \end{pmatrix}$

$$\Gamma = \frac{1}{\sqrt{1 - \frac{\vec{A}^2}{c^2}}}$$

$\updownarrow$ inverse

$$\Lambda^{-1A}{}_B = \begin{pmatrix} \Gamma & -\frac{\Gamma A_c}{c} \\ -\Gamma \frac{A^b}{c} & \delta^b{}_c + \frac{A^b A_c}{\vec{A}^2}(\Gamma - 1) \end{pmatrix}$$

$$\boxed{\begin{aligned} v' &= \Gamma\left(v + A^a e_a\right) \\ &= \Gamma\left(v + \vec{A}\right) \\ e'_b &= v\frac{\Gamma A_b}{c^2} + e_b + \frac{\vec{A}\, A_b}{\vec{A}^2}(\Gamma - 1) \\ &\Big\downarrow c \to \infty \\ v' &= v + \vec{A} \\ e'_a &= e_a \end{aligned}}$$

$$\theta'^A = \Lambda^{-1A}{}_B\, \theta^B \qquad \vec{A}^2 = A^a A^b \delta_{ab} = A^i A^j a_{ij}$$

$$\boxed{\begin{aligned} \tau' &= \Gamma\left(\tau - \frac{A_c \theta^c}{c^2}\right) = \Gamma\left(\tau - \frac{\vec{A}}{c^2}\right) \\ \theta'^b &= \theta^b - \Gamma A^b \tau + \frac{A^b \vec{A}}{\vec{A}^2}(\Gamma - 1) \\ &\Big\downarrow c \to \infty \\ \tau' &= \tau \qquad \theta'^b = \theta^b - A^b \tau \\ &\qquad\qquad\quad\;\; = \theta^b + A^b \mu \end{aligned}}$$

Galilean group:
* invariant clock form
* shift in the field of observers

Going back to the Zermelo Galilean parameterization where

$$v = \frac{1}{\Omega}(\partial_t + w^i \partial_i) \qquad \hat{dx}^i = dx^i - w^i dt \qquad \mu = -\Omega dt$$

$$\left.\begin{aligned} v' &= \frac{1}{\Omega}(\partial_t + w^i \partial_i) + A^i \partial_i \\ \hat{dx}'^i &= \hat{dx}^i + A^i(-\Omega dt) = dx^i - w^i dt - A\Omega dt \end{aligned}\right\} \qquad \boxed{\begin{array}{c} \frac{w^i}{\Omega} \\ \downarrow \\ \frac{w'^i}{\Omega} = \frac{w^i}{\Omega} + A^i \end{array}}$$

Note: This is not a coordinate transformation but a local galilean boost

§3. Connections

A Newton-Cartan manifold can be equipped with a connection, i.e. a parallel transport. It is then called a Newton-Cartan or a Galilean structure. This structure is strong if it is metric and clock-form invariant.

Reminder on connections in general

We start with the data $\{\theta^A\} \leftrightarrow \{e_A\}$, which carry an important information stored in

$$\boxed{\begin{array}{c} d\theta^A = \frac{1}{2} C^A{}_{BC}\, \theta^C \wedge \theta^B \\ \Updownarrow \\ [e_A, e_B] = C^C{}_{AB}\, e_C \end{array}}$$

A connection is a rule for parallel transport, encoded as follows

$$\nabla_{e_A} \theta^B = -\Gamma^B{}_{AC}\, \theta^C \Leftrightarrow \nabla_{e_A} e_B = \Gamma^C{}_{AB}\, e_C$$

The connection one-form reads $\omega^A{}_B = \Gamma^A{}_{CB}\, \theta^C$

The torsion two-form is $\mathcal{T}^A = d\theta^A + \omega^A{}_C \wedge \theta^C = \frac{1}{2} T^A{}_{BC}\, \theta^B \wedge \theta^C$

The curvature two-form is $\mathcal{R}^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B = \frac{1}{2} R^A{}_{BCD}\, \theta^C \wedge \theta^D$

The latter obey integrability conditions leading to Bianchi identities in the instance of Levi-Civita connection

The covariant derivative of the cometric g^{AB} reads — if a cometric is available as $\partial_s^2 = g^{AB} e_a e_b$

$$\theta^c \nabla_c g^{AB} = dg^{AB} + \omega^{AB} + \omega^{BA}$$ and this defines the non-metricity one-form

<u>Back to the Newton Cartan</u> with "hats" to distinguish it from the Lorentzian ascendent, if any. The forms are $\{\tau, d\hat{x}^i\}$ and vectors $\{v, \partial_i\}$. $\tau = \Omega dt$

$$\boxed{-d\mu = d\tau = \varphi_i \, d\hat{x}^i \wedge \tau - \varpi_{ij} \, d\hat{x}^i \wedge d\hat{x}^j}$$

$d\hat{x}^i = dx^i - w^i dt$

$v = \frac{1}{\Omega}(\partial_t + w^i \partial_i)$

$$\boxed{d\,d\hat{x}^i = \frac{1}{\Omega} \partial_k w^i \quad \tau \wedge d\hat{x}^k = C^i{}_{kt} \, \tau \wedge d\hat{x}^k}$$

So for the connection we obtain

$$\left.\begin{aligned}
\hat{\nabla}_v \tau &= -\hat{\Gamma}^t{}_{tt} \, \tau - \hat{\Gamma}^t{}_{ti} \, d\hat{x}^i \overset{!}{=} 0 \\
\hat{\nabla}_{\partial_i} \tau &= -\hat{\Gamma}^t{}_{it} \, \tau - \hat{\Gamma}^t{}_{ij} \, d\hat{x}^j \overset{!}{=} 0
\end{aligned}\right\} \text{because of the strong hypothesis on } \tau$$

$$\begin{aligned}
\hat{\nabla}_v d\hat{x}^i &= -\hat{\Gamma}^i{}_{tt} \, \tau - \hat{\Gamma}^i{}_{tj} \, d\hat{x}^j = -\delta^i \tau - \hat{\gamma}^i{}_j \, d\hat{x}^j \\
\hat{\nabla}_{\partial_i} d\hat{x}^k &= -\hat{\Gamma}^k{}_{it} \, \tau - \hat{\Gamma}^k{}_{ij} \, d\hat{x}^j = -\beta^k{}_i \tau - \hat{\gamma}^k{}_{ij} \, d\hat{x}^j
\end{aligned}$$

As a consequence $\hat{\nabla}_v \partial_i = \hat{\gamma}^j{}_i \partial_j \quad \hat{\nabla}_{\partial_i} \partial_j = \hat{\gamma}^k{}_{ij} \partial_k$

$\hat{\nabla}_v v = \delta^i \partial_i \quad \hat{\nabla}_{\partial_i} v = \beta^j{}_i \partial_j$

Before imposing the metric compatibility some remarks are in order.

- Under Galilean diffeomorphisms $(t,\vec{x}) \to (t'(t), \vec{x}'(t,x))$

υ and τ are invariant, and ∂_i and $\hat{dx}^j$ transform homogeneously: $\partial_i' = J^{-1j}{}_i \partial_j \quad \tilde{dx}^j = J^j{}_i \hat{dx}^i$

where $J^j{}_i = \frac{\partial x'^j}{\partial x^i}$, with the consequence: tensors transform as Galilean tensors

φ_i a_{ij}

$\Rightarrow \delta^i$ and $\beta^i{}_j$ transform as Galilean tensors

$\Rightarrow \hat{\gamma}^i{}_j$ and $\hat{\gamma}^k{}_{ij}$ " as " connections

- As we will see in the next chapter δ^i and $\beta^i{}_j$ are not present when considering the limit of a Levi-Civita connection of the Zermelo Lorentzian ascendent

$$ds^2 = -c^2 \frac{\Omega^2 dt^2}{c^2} + a_{ij} \hat{dx}^i \hat{dx}^j$$

- The Galilean tensors $\beta^i{}_j$ and δ^i spoil the time-space splitting. Consider for that a general vector field on the Newton-Cartan manifold: $V = V^t \upsilon + V^i \partial_i$

With respect to Galilean diffeomorphisms V^t is a scalar and V^i is a vector and the vector V is indeed reduced. What about $\hat{\nabla}_\upsilon V$ and $\hat{\nabla}_{\partial_i} V$?

$$\hat{\nabla}_\upsilon V = \hat{\nabla}_\upsilon \left(V^t \upsilon + V^i \partial_i\right) = \upsilon(V^t)\upsilon + V^t \delta^i \partial_i + \upsilon(V^i)\partial_i + V^i \hat{\gamma}^j{}_i \partial_j$$
$$= \hat{\nabla}_t V^t \upsilon + \hat{\nabla}_t V^i \partial_i$$

So we find
$$\left.\begin{aligned} \hat{\nabla}_t V^t &= \frac{1}{\Omega} \hat{D}_t V^t = \upsilon(V^t) = \frac{1}{\Omega} \hat{\partial}_t V^t \\ \hat{\nabla}_t V^i &= \frac{1}{\Omega} \hat{D}_t V^i = \upsilon(V^i) + \hat{\gamma}^i{}_j V^j + \underline{\delta^i V^t} \end{aligned}\right\}$$

(In Lorentz: $\nabla_{\partial_\mu} V = \nabla_\mu V^\nu \partial_\nu = \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho\right)\partial_\nu$

$$\hat{\nabla}_{\partial_i} V = \hat{\nabla}_{\partial_i}\left(V^t v + V^j \partial_j\right) = \partial_i V^t v + V^t \beta^j{}_i \partial_j + \partial_i V^j \partial_j + V^j \hat{\gamma}^k{}_{ij} \partial_k = \hat{\nabla}_i V^t v + \hat{\nabla}_i V^j \partial_j$$

with
$$\begin{cases} \hat{\nabla}_i V^t = \partial_i V^t \\ \hat{\nabla}_i V^j = \partial_i V^j + \hat{\gamma}^j{}_{ik} V^k + \underline{\beta^j{}_i V^t} \end{cases}$$

And similarly for general forms: $W = W_t \tau + W_i \hat{dx}^i$

we show (exercise)
$$\hat{\nabla}_t W_t = \frac{1}{\Omega} \hat{D}_t W_t = \frac{1}{\Omega}\hat{\partial}_t W_t \underline{- \delta^i W_i}$$
$$\hat{\nabla}_t W_i = \frac{1}{\Omega} \hat{D}_t W_i = \frac{1}{\Omega}\hat{\partial}_t W_i - \hat{\gamma}^j{}_i W_j$$
$$\hat{\nabla}_i W_t = \partial_i W_t \underline{- \beta^j{}_i W_j}$$
$$\hat{\nabla}_i W_j = \partial_i W_j - \hat{\gamma}^k{}_{ij} W_k$$

- We will set $\beta^i{}_j = 0$ and $\delta^i = 0$ to preserve the time and space splitting, and since they are Galilean tensors this is legitimate. However we will come back later to their role in restoring the Galilean boost invariance, usually lost in ordinary fluids.

Torsion and non-metricity

Let's start with the connection one-form:

$$\hat{\omega}^i{}_t = \hat{\Gamma}^i{}_{tt}\,\tau + \hat{\Gamma}^i{}_{jt}\,d\hat{x}^j = \delta^i\,\tau + \beta^i{}_j\,d\hat{x}^j$$

$$\hat{\omega}^t{}_t = 0 \qquad \hat{\omega}^t{}_i = 0 \qquad \text{(because the structure is strong)}$$

$$\hat{\omega}^i{}_j = \hat{\Gamma}^i{}_{tj}\,\tau + \hat{\Gamma}^i{}_{kj}\,d\hat{x}^k = \hat{\gamma}^i{}_j\,\tau + \hat{\gamma}^i{}_{\ell j}\,d\hat{x}^\ell$$

We will not discuss the curvature, but the torsion, using first Cartan's equation:

$$\hat{\mathcal{T}}^t = d\tau + \hat{\omega}^t{}_A \wedge \theta^A = d\tau \quad \text{(the second term because "strong")}$$

so using previous data we find $\boxed{\hat{\mathcal{T}}^t = \underbrace{\varphi_i}_{=\partial_i \ln\Omega}\, d\hat{x}^i \wedge \tau - \underbrace{\omega_{ij}}_{=0 \text{ here}}\, d\hat{x}^i \wedge d\hat{x}^j}$

Note: a strong Newton-Cartan structure is called «torsionless» in the literature if $\hat{\mathcal{T}}^t = 0$ i.e. if τ is exact, which amounts to having a universal Newtonian time. This is a misnomer however because the torsion has other components.

$$\begin{aligned}
\hat{\mathcal{T}}^j &= d\,d\hat{x}^j + \hat{\omega}^j{}_t \wedge \tau + \hat{\omega}^j{}_i \wedge d\hat{x}^i \\
&= \underbrace{\frac{1}{\Omega}\partial_k w^j}_{C^j{}_{kt}}\, \tau \wedge d\hat{x}^k + \beta^j{}_i\, d\hat{x}^i \wedge \tau + \hat{\gamma}^j{}_i\, \tau \wedge d\hat{x}^i \\
&\qquad + \hat{\gamma}^j{}_{ki}\, d\hat{x}^k \wedge d\hat{x}^i \\
&= \left(\frac{1}{\Omega}\partial_k w^j - \beta^j{}_k + \hat{\gamma}^j{}_k\right) \tau \wedge d\hat{x}^k + \hat{\gamma}^j{}_{ki}\, d\hat{x}^k \wedge d\hat{x}^i \\
&= \hat{T}^j{}_{tk}\, \tau \wedge d\hat{x}^k + \frac{1}{2}\hat{T}^j{}_{ki}\, d\hat{x}^k \wedge d\hat{x}^i
\end{aligned}$$

summarizing

$$\boxed{\hat{\mathcal{T}}^j \rightarrow \begin{array}{l} \hat{T}^j{}_{tk} = \frac{1}{\Omega}\partial_k w^j + \hat{\gamma}^j{}_k - \beta^j{}_k \\ \hat{T}^j{}_{ik} = 2\,\hat{\gamma}^j{}_{[ik]} \end{array}}$$

For the next steps we will consider $\beta^j{}_k = 0$, as we already stated, as well as $d\tau = 0$ and $\hat{\gamma}^j{}_{[ik]} = 0$. This ensures the existence of a universal time ($\Omega = \Omega(t)$) but $\hat{\mathcal{T}}^j$ keeps a non-vanishing piece (the connection is not utterly torsion-free)

We will now handle the metric compatibility. The metric is usually spelled $ds^2 = q^{AB}\, e_A e_B = a^{ij}\, \partial_i \partial_j = dl^2$ and demanding $\theta^C \nabla_C q^{AB} = 0$ gives

- $0 = da^{ij} + 2\hat{\omega}^{(ij)} \Leftrightarrow \left(\frac{1}{\Omega}\hat{\partial}_t a^{ij} + 2\hat{\gamma}^{(ij)}\right)\tau + \left(\partial_k a^{ij} + \hat{\gamma}^{(i}{}_k{}^{j)}\right) d\hat{x}^k = 0$
- $0 = \hat{\omega}^{tt}$
- $0 = \hat{\omega}^{(ti)}$ $\Big\}$ automatic $\quad (q^{ti} = q^{tt} = 0,\ \hat{\omega}^t{}_t = \hat{\omega}^t{}_i = 0)$

Metric compatibility requires $\boxed{\hat{\gamma}^{(ij)} = -\frac{1}{2\Omega}\hat{\partial}_t a^{ij}}$

and, together with $\hat{\gamma}^i{}_{[jk]} = 0$ $\quad \boxed{\hat{\gamma}^i{}_{jk} = \frac{1}{2} a^{il}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right)}$

Notice that the torsion reads, for the remaining piece of $\hat{\mathcal{T}}^j$:

$$
\begin{aligned}
\hat{T}^j{}_{tk} &= \frac{1}{\Omega}\partial_k w^j + \hat{\gamma}^{(jl)} a_{lk} + \hat{\gamma}^{[jl]} a_{lk} - \beta^j{}_k \\
&= \frac{1}{\Omega}\partial_k w^j - \frac{1}{2\Omega}\hat{\partial}_t a^{jl} a_{lk} - \beta^{(jl)} a_{lk} + \hat{\gamma}^{[jl]} a_{lk} - \beta^{[jl]} a_{lk} \\
&= a_{lk}\left[\frac{1}{\Omega} a^{i(l}\partial_i w^{j)} - \frac{1}{2\Omega}\hat{\partial}_t a^{jl} - \beta^{(jl)}\right] \\
&\quad + a_{lk}\left[\frac{1}{\Omega} a^{i[l}\partial_i w^{j]} + \hat{\gamma}^{[jl]} - \beta^{[jl]}\right]
\end{aligned}
$$

In summary

$$
\boxed{\hat{\mathcal{T}}^j = \left[a_{lk}\left(\frac{1}{\Omega} a^{i(l}\partial_i w^{j)} - \frac{1}{2\Omega}\hat{\partial}_t a^{jl} - \beta^{(jl)}\right) + a_{lk}\left(\frac{1}{\Omega} a^{i[l}\partial_i w^{j]} + \hat{\gamma}^{[jl]} - \beta^{[jl]}\right)\right]\tau \wedge d\hat{x}^k}
$$

In conclusion, for a strong connection with $\hat{T}^k_{ij} = 0$ the parts of the connection which are not determined by the geometry (ie by Ω, w^i and a^{ij}) are (11)

δ^i	d
β^{ij}	d^2
$\hat{\gamma}^{[ij]}$	$\frac{d(d-1)}{2}$

<u>Options</u> — besides setting $d\tau = 0$ or keeping φ_i and ω_{ij}

$$\boxed{\hat{\gamma}^j = \hat{\gamma}^{wj}{}_k \, \tau \wedge \hat{dx}^k} \Leftarrow$$

- ours — because descendent from Lorentz

$$\underline{\delta^i = \beta^{ij} = 0} \qquad \underline{\hat{\gamma}_{[ij]} = -\frac{1}{\Omega} a_{k[i}\partial_{j]} w^k}$$

— with a symmetric Galilean tensor — but in this instance, the torsion does not vanish

$$\boxed{\hat{\gamma}^w{}_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij} + \hat{\nabla}_{(i} w_{j)}\right)}$$

- a rather popular (since Duval 1978) option is to require only that the torsion vanishes (besides setting $d\tau = 0 \Rightarrow \hat{\mathcal{T}}^t = 0$)

This implies that

$$\begin{cases} \beta^{(jl)} = \underbrace{\frac{1}{\Omega} a^{i(l}\partial_i w^{j)} - \frac{1}{2\Omega}\hat{\partial}_t a^{jl}}_{\text{this is a Galilean tensor}} = \hat{\gamma}^{wjl} \\ \hat{\gamma}^{[jl]} = \frac{1}{\Omega} a^{i[j}\partial_i w^{l]} + \beta^{[jl]} \quad \text{(not a Galilean tensor)} \end{cases}$$

So we are left with

$\beta^{[jl]}$	$\frac{d(d-1)}{2}$
both Galilean tensors δ^i	d
	$\frac{d(d+1)}{2}$

This defines a spacetime 2-form

$$\boxed{\begin{aligned} F &= \delta_k \, \tau \wedge \hat{dx}^k - \beta_{ij} \, \hat{dx}^i \wedge \hat{dx}^j \\ &= \hat{\omega}_{it} \wedge \hat{dx}^i \end{aligned}}$$

§4. Galilean hydrodynamics from Lorentzian relativity

We start with $ds^2 = -c^2\Omega^2 dt^2 + a_{ij}(dx^i - w^i dt)(dx^j - w^j dt)$
assuming, as already mentioned, that no other dependence in c^2 exists, and compute $\nabla_\mu T^{\mu\nu}$ in the limit $c \to \infty$

More precisely we will consider $c\Omega \nabla_\mu T^{\mu 0}$, which is a scalar under Galilean diffeomorphisms and $\nabla_\mu T^\mu{}_i$, which is a one form " " "

We find actually $c\Omega \nabla_\mu T^{\mu 0} = c^2 \mathcal{C} + \mathcal{E} + o(1/c^2)$

∇_μ : Levi-Civita (Ω, w^i, a_{ij}) $\qquad \nabla_\mu T^\mu{}_i = \mathcal{M}_i + o(1/c^2)$

but for that, it has been necessary to assume the behavior of the various observables which enter the energy-momentum tensor.

$$T^{\mu\nu} = \varepsilon \frac{u^\mu u^\nu}{c^2} + p\left(\frac{u^\mu u^\nu}{c^2} + g^{\mu\nu}\right) + \tau^{\mu\nu} + \frac{q^\mu u^\nu + q^\nu u^\mu}{c^2}$$

$$u = \gamma(\partial_t + v^i \partial_i) = \gamma\left(\hat{\partial}_t + (v^i - w^i)\partial_i\right)$$
$$= \gamma\Omega\left(\frac{1}{\Omega}\hat{\partial}_t + \frac{v^i - w^i}{\Omega}\partial_i\right)$$

property : $\frac{1}{\Omega}\hat{\partial}_t$ is geodesic $\longrightarrow$

$\boxed{\alpha^i = \frac{v^i - w^i}{\Omega} \text{ will be velocity wrt a local inertial frame}}$

is a Galilean vector
(vector under Galilean diffeos)

normalization $\|u\|^2 = -c^2$

$$\boxed{\gamma\Omega = \frac{1}{\sqrt{1 - \frac{\vec{\alpha}^2}{c^2}}} \qquad \vec{\alpha}^2 = a_{ij}\alpha^i\alpha^j}$$

So $\quad u = \left(v + \alpha^i \partial_i\right) + \frac{\vec{\alpha}^2}{2c^2}\left(v + \alpha^i \partial_i\right) + \sigma(1/c^4)$

and we also have: $\quad \mathcal{E} = (c^2 + e)\, \rho_0$

rest specific energy — internal specific energy — relativistic rest mass density

q^μ and $\tau^{\mu\nu}$ are transverse, so fully determined by their spatial components τ_{ij} and q_i with behaviours wrt c^2

Important remark: actually the presence of ρ_0 betrays the existence of a conserved current (e.g. baryon number):

$$\boxed{\nabla_\mu J^\mu = 0 \qquad J^\mu = \rho_0 u^\mu + \underbrace{j^\mu}_{\text{transverse}}}$$

and we must assume the behaviour at large c^2. To make the long story short and focus on the energy-momentum exclusively we have the following behaviours:

$$\begin{cases} q_i = Q_i + \sigma(1/c^2) \\ \rho_0 = \rho - \frac{1}{c^2}\left(\frac{\rho}{2}\vec{\alpha}^2\right) + \sigma(1/c^4) \qquad \left(\rho_0 = \frac{\rho}{\gamma\Omega}\right) \\ \tau_{ij} = -\Sigma_{ij} \end{cases}$$

So that $\quad \begin{cases} \mathcal{E} = \rho c^2 + \rho\left(e - \frac{1}{2}\vec{\alpha}^2\right) + \sigma(1/c^2) \\ p = P \end{cases}$

We find, putting all pieces together

the following equations:

$$\boxed{\begin{aligned} 0 &= \mathcal{C} = \frac{1}{\Omega}\hat{\partial}_t \rho + \theta^w \rho + \hat{\nabla}_i P^i \\ 0 &= \mathcal{E} = \frac{1}{\Omega}\hat{\partial}_t \Pi + \theta^w \Pi + \Pi_{ij}\hat{\gamma}^{wij} + \hat{\nabla}_i \Pi^i \\ 0 &= \mathcal{M}_i = \frac{1}{\Omega}\hat{D}_t P_i + \theta^w P_i + P_j \hat{\gamma}^{wj}{}_i + \hat{\nabla}^j \Pi_{ji} \end{aligned}}$$

Continuity

Energy

Euler

where $\hat{D}_t$, $\hat{\nabla}_i$ are the time and space Galilean connections that we introduced, fully determined by the limiting Newton-Cartan geometry with $\boxed{\hat{T}^j{}_{tk} = \hat{\gamma}^{wj}{}_k}$ $\boxed{\theta^w = \hat{\gamma}^{wk}{}_k}$

$\rightsquigarrow$ symmetric Galilean tensor.

with the Galilean momenta:

$P^i = \rho \alpha^i$ matter current (or simply momentum)

$h = e + \frac{p}{\rho}$ enthalpy

$\Pi_i = \rho \alpha_i \left(h + \frac{1}{2}\vec{\alpha}^2\right) - \underset{\substack{\uparrow \\ \text{viscosity-related}}}{\alpha^j \Sigma_{ji}} + \underset{\substack{\uparrow \\ \text{heat current}}}{Q_i}$ energy flux

$\Pi_{ij} = \rho \alpha_i \alpha_j + p a_{ij} - \Sigma_{ij}$ energy-stress tensor

$\Pi = \rho\left(e + \frac{1}{2}\vec{\alpha}^2\right)$ fluid energy density

These are the standard fluid equations on general Newton-Cartan with curvature but with a connection inherited from a Lorentzian Levi-Civita.

Under the carpet

- precise role of the extra $U(1)$ current which is here identified with the matter current through the continuity equation

- contribution of the extra $U(1)$ to the galilean heat current Q_i
- fate of the hydrodynamic-frame invariance, absent here, but potentially present if
  - * the elementary carriers are nonlon
  - * there is a sink or a spring producing loss or gain of matter: non conservation

- we could have kept τ general with

$$d\tau = \varphi_i \, \hat{dx}^i \wedge \tau - \omega_{ij} \, \hat{dx}^i \wedge \hat{dx}^j \Rightarrow \hat{\tau}^t$$

$$\square \quad \nabla_\mu T^\mu{}_i = c^2 \mathcal{N}_i + \mathcal{M}_i + \sigma(1/c^2)$$

$$\square \quad \mathcal{C} \to \quad + 2\varphi_i P^i$$

$$\mathcal{E} \to \quad + 2\varphi_i \Pi^i$$

$$\mathcal{M}_i \to \quad + \varphi_i \Pi + \varphi^j \Pi_{ji} + 2\, \Pi^j \omega_{ji}$$

$$\mathcal{N}_i = \varphi_i \varrho + 2 P^j \omega_{ji}$$

$$\mathcal{N}_i = 0 \iff \begin{cases} * \text{ Either } \varrho = 0 \\ * \text{ or } d\tau = 0 \end{cases}$$

§ 5. Variational principle, extra degrees of freedom, boosts

Geometry (dτ=0 & our geometry)

The momenta $\Pi, \Pi_i, P_i, \Pi_{ij}$ are variations of $S=\int \underbrace{dt\, d^dx \sqrt{a}\,\Omega}_{\text{volume form in Galilean Zermelo frame for a Newton Cartan geometry}} \mathcal{L}$ $\qquad S=S[a^{ij}, w^i, \Omega]$

$$\Pi_{ij} = \frac{-2}{\sqrt{a}\,\Omega}\frac{\delta S}{\delta a^{ij}} \qquad P_i = -\frac{1}{\sqrt{a}}\frac{\delta S}{\delta w^i} \qquad \Pi = -\frac{1}{\sqrt{a}}\frac{\delta S}{\delta \Omega}$$

$$\delta S = -\int dt\,\Omega \int d^dx \sqrt{a}\left(\frac{1}{2}\Pi_{ij}\,\delta a^{ij} + \frac{P_i}{\Omega}\delta w^i + \frac{\Pi}{\Omega}\delta\Omega\right)$$

Imposing $\delta_\xi S = 0$ where $\xi = \xi^t \upsilon + \xi^i \partial_i$ with $\xi^t = \xi^t(t)$ because this is a Galilean-diffeomorphism generator and $\Omega=\Omega(t)$

and
$$-\delta_\xi a^{ij} = \mathcal{L}_\xi a^{ij} = -2\left(\hat{\nabla}^{(i}\xi^{j)} + \hat{\gamma}^{w\,ij}\,\xi^t\right)$$
$$-\delta_\xi \Omega = \mathcal{L}_\xi \Omega = \partial_t \xi^t + \mathcal{L}_{\vec{w}}\xi^t$$
$$-\delta_\xi w^i = \mathcal{L}_\xi w^i = -\partial_t \xi^i - \mathcal{L}_{\vec{w}}\xi^i$$

$$\delta_\xi S = \int dt\, d^dx \sqrt{a}\,\Omega\left\{-\xi^t\left(\frac{1}{\Omega}\hat{\partial}_t \Pi + \theta^w \Pi + \Pi_{ij}\hat{\gamma}^{w\,ij}\right) + \xi^i\left(\frac{1}{\Omega}\hat{D}_t P_i + P_j \hat{\gamma}^{w\,j}{}_i + \hat{\nabla}^j \Pi_{ij} + \theta^w P_i\right)\right\} + \text{b.t.}$$

$$\Leftrightarrow \boxed{\begin{aligned} &\frac{1}{\Omega}\hat{\partial}_t \Pi + \theta^w \Pi + \Pi_{ij}\hat{\gamma}^{w\,ij} = \underline{-\hat{\nabla}_i \Pi^i} \\ &\left(\frac{1}{\Omega}\hat{D}_t + \theta^w\right)P_i + P_j\hat{\gamma}^{w\,j}{}_i + \hat{\nabla}^j \Pi_{ij} = 0 \end{aligned}} \quad \begin{aligned} &\to \mathcal{E} \\ &\to \mathcal{M}_i \end{aligned}$$

Gauge field

Suppose that $S = S[\hat{B}, B_i]$ $\qquad \mathcal{B} = \hat{B}\tau + B_i d\hat{x}^i$

with gauge invariance under $\delta_\Lambda \mathcal{B} = d\Lambda$ $\begin{cases} \delta_\Lambda \hat{B} = -\frac{1}{\Omega}\hat{\partial}_t \Lambda \\ \delta_\Lambda B_i = -\partial_i \Lambda \end{cases}$

$$\delta S = \int dt\, d^dx \sqrt{a}\,\Omega \left(\rho\, \delta\hat{B} + N^i \delta B_i \right) \text{ where } \quad \rho = \frac{1}{\sqrt{a}\,\Omega}\frac{\delta S}{\delta \hat{B}}$$

$$N^i = \frac{1}{\sqrt{a}\,\Omega}\frac{\delta S}{\delta B^i}$$

$$0 = \delta_\Lambda S = -\int dt\, d^dx \sqrt{a} \left(\rho\, \hat{\partial}_t \Lambda + \Omega N^i \partial_i \Lambda \right)$$

$$= \int dt\, d^dx\, \Omega \sqrt{a}\, \Lambda \left(\left(\tfrac{1}{\Omega}\hat{\partial}_t + \theta^w\right)\rho + \hat{\nabla}_i N^i \right) + \text{b.t.}$$

$$\Leftrightarrow \quad \boxed{\left(\frac{1}{\Omega}\hat{\partial}_t + \theta^w\right)\rho + \hat{\nabla}_i N^i = 0} \quad \leftrightarrow \mathcal{C}$$

Questions

* Where could this field come from?
* What is its contribution in $\delta_\xi S$ and to the conservation equations $\mathcal{E}$ and $\mathcal{M}_i$?
* Eventually $S = S[a^{ij}, w^i, \Omega, \hat{B}, B_i]$, so how do a^{ij}, w^i, Ω behave under δ_Λ? And how does this affect the conservation equation $\mathcal{C}$?

* Finally comes the question of invariance under Galilean boosts. Remember that an arbitrary Galilean vector with $A^i = A^i(t, \vec{x})$ prod

$$\begin{cases} v' = v + \vec{A} \\ \partial'_i = \partial_i \end{cases} \qquad \begin{cases} \tau' = \tau \\ d\hat{x}'^i = d\hat{x}^i + A^i \Omega\, dt \\ \Omega' = \Omega \end{cases} \qquad w'^i = w^i + \Omega A^i$$

Infinitesimally $\vec{A} = \delta v$

$$\boxed{P_i \neq 0 \Leftrightarrow \text{breaking of Galilean boost invariance}} \qquad \Updownarrow \text{ since } \frac{\delta S}{\delta w^i} = -\sqrt{a}\, P_i$$

* Use <u>Duval's connection</u> which has no torsion at all thanks to $\beta^{(jl)} = \hat{\gamma}^{wjl}$ and keep $\beta^{[ij]}, \delta^i$ as extra degrees of freedom. These will have associated <u>momenta</u> canonically conjugate to $\underline{F = \delta_k \, \tau \wedge \hat{dx}^k - \beta_{ij} \, \hat{dx}^i \wedge \hat{dx}^j}$

* Restrict the Duval connection to the so-called <u>Newtonian connection</u> which obeys further $\boxed{dF = 0}$

Hence the relevant degree of freedom is now

$$\underline{\mathcal{B} = \hat{B}\,\tau + B_i \, \hat{dx}^i}$$

$$\boxed{\begin{array}{l} \mathcal{B} \text{ such that } F = d\mathcal{B} \\ \text{with a } \underline{\text{built-in gauge invariance}} \\ \qquad \mathcal{B} \to \mathcal{B} - d\Lambda \end{array}}$$

In total the degrees of freedom are

$$\{ a^{ij}, \; w^i, \; \Omega, \; \hat{B}, \; B_i \}$$

and their dual momenta

$$\{ \Pi_{ij}, \; P_i, \; \Pi, \; \varrho, \; N^i \}$$

with
$$\begin{cases} \delta_\Lambda a^{ij} = \delta_\Lambda w^i = \delta_\Lambda \Omega = 0 \\ \delta_\Lambda \hat{B} = -\frac{1}{\Omega} \hat{\partial}_t \Lambda \quad \delta_\Lambda B_i = -\partial_i \Lambda \end{cases}$$

This will provide the usual conservation equation but there is an inflation of the number of degrees of freedom: P_i and N^i are a priori independent since $\frac{\delta}{\delta w^i}$ and $\frac{\delta}{\delta B_i}$ are.

* In order to reduce appropriately the number of degrees of freedom we can further require the restoration of invariance under local Galilean boosts.

This will set a relationship amongst P_i and N^i.

Calling δ_A the local Galilean boost transformation we must find $\delta_A a^{ij}$, $\delta_A w^i$, $\delta_A \Omega$, $\delta_A \hat{B}$ and $\delta_A B_i$.

The guideline is the requirement of invariance of the basic geometric data of the strong Newton-Cartan structure: the cometric a^{ij}, the clock form Ω and

Concrete process $\qquad \delta_A v = \vec{A} \qquad \delta_A \tau = 0 \qquad \delta_A a^{ij} = \delta_A \Omega = 0 \quad \delta_A \partial_i = 0$

$$= \delta_A \vec{w} = \Omega \vec{A} \qquad \delta_A d\hat{x}^i = -A^i \Omega\, dt = -A^i \tau$$

$\left(\begin{array}{l}\text{reminder:} \\ d\hat{x}^i = dx^i - w^i dt\end{array} \quad v = \frac{1}{\Omega}(\partial_t + w^i \partial_i) \quad \tau = \Omega\, dt\right)$

* Transformation of the connection

reminder: $\quad \hat{\nabla}_v v = \delta^i \partial_i \qquad \hat{\nabla}_{\partial_i} v = \beta^j{}_i \partial_j$

$$\hat{\nabla}_v \partial_i = \hat{\gamma}^j{}_i \partial_j \qquad \hat{\nabla}_{\partial_i} \partial_j = \hat{\gamma}^k{}_{ij} \partial_k$$

and the transformed connection coefficient will be read off in

$$\hat{\nabla}_{v'} v' = \delta'^i \partial_i' \qquad \hat{\nabla}_{\partial_i'} v' = \beta'^j{}_i \partial_j'$$

$$\hat{\nabla}_{v'} \partial_i' = \hat{\gamma}'^j{}_i \partial_j' \qquad \hat{\nabla}_{\partial_i'} \partial_j' = \hat{\gamma}'^k{}_{ij} \partial_k'$$

Applying the basic rules (linearity wrt the vector and Leibniz) we find

$$\boxed{\begin{array}{ll}\delta_A \hat{\gamma}^k{}_{ij} = 0 & \hat{\delta}_A \hat{\gamma}^j{}_i = A^k \hat{\gamma}^j{}_{ki} \\ \delta_A \beta^j{}_i = \hat{\nabla}_i A^j & \delta_A \delta^j = A^i(\beta^j{}_i + \hat{\gamma}^j{}_i) \\ & \quad + v(A^j) \\ & = A^i \beta^j{}_i + \frac{1}{\Omega} \hat{D}_t A^j\end{array}}$$

Remark on the torsion $\quad \delta_A \tau = 0 \Rightarrow \delta_A d\tau = 0$

$\Rightarrow \delta_A \hat{\tau}^t = 0 \qquad ([\delta_A, d] = 0)$

$$\delta_A \tau^i = \delta_A \left(\hat{T}^i{}_{tj}\, \tau \wedge d\hat{x}^j + \frac{1}{2} \hat{T}^i{}_{kj}\, d\hat{x}^k \wedge d\hat{x}^j \right)$$

$$= \delta_A \hat{T}^i{}_{tj}\, \tau \wedge d\hat{x}^j + \hat{T}^i{}_{tj} \underbrace{\tau \wedge \delta_A d\hat{x}^j}_{=0} + \underbrace{\frac{1}{2} \delta_A \hat{T}^i{}_{kj}\, d\hat{x}^k \wedge d\hat{x}^j}_{=0} + \underbrace{\hat{T}^i{}_{kj}}_{=0}\, \delta_A d\hat{x}^k \wedge d\hat{x}^j$$

$\hat{T}^i{}_{tj} = \hat{\gamma}^i{}_j - \beta^i{}_j - C^i{}_{tj}$

$\delta_A \hat{T}^i{}_{kj} = 2\delta_A \hat{\gamma}^i{}_{[kj]} = 0$

car $\hat{\gamma}^i{}_{kj} = 0$ $\qquad$ car $\delta \hat{\gamma}^i{}_{kj} = 0$

$\delta_A \hat{T}^i{}_{tj} = \delta_A \hat{\gamma}^i{}_j - \delta_A \beta^i{}_j + \partial_j A^i$

$= A^k \hat{\gamma}_k{}^i{}_j - \hat{\nabla}_j A^i + \partial_j A^i$

$= 0$

$C^i{}_{tj} = -\frac{\Lambda}{\Omega} \partial_j w^i$

$\delta_A C^i{}_{tj} = -\partial_j A^i$

$= 0$

$\Rightarrow$ Invariance of the torsion (remains zero)

* Transformation of $F = \delta_k\, \tau \wedge d\hat{x}^k - \beta_{ij}\, d\hat{x}^i \wedge d\hat{x}^j$

$$\delta_A F = \delta_A \delta_k\, \tau \wedge d\hat{x}^k - \delta \hat{\beta}_{ij}\, d\hat{x}^i \wedge d\hat{x}^j + 2\hat{\beta}_{(ij)} A^i\, \tau \wedge d\hat{x}^j$$

After a straightforward computation we find

$$\boxed{\delta_A F = d\left(A_i\, d\hat{x}^i\right)} \qquad A_i = a_{ij} A^j$$

Since $F = d\mathcal{B}$ and $[\delta_A, d] = 0$ $\qquad \boxed{\delta_A \mathcal{B} = A_i\, d\hat{x}^i - d\Lambda}$

for a Newtonian connection

$$\Rightarrow \boxed{\begin{array}{ll} \delta_A \hat{B} = B_i A^i - \frac{1}{\Omega} \partial_t \Lambda & \delta_A B_i = A_i - \partial_i \Lambda \\ \text{où} \quad \mathcal{B} = \hat{B}\, \tau + B_i\, d\hat{x}^i & \end{array}}$$

Back to geometry and gauge field

Finally, using the Duval-Newton connection the geometric data are $\{a^{ij}, w^i, \Omega, \hat{B}, B_i\}$ and general the variation of the action reads:

$$\boxed{\delta S = -\int dt\, \Omega \int d^d x \sqrt{a} \left(\frac{1}{2} \Pi_{ij}\, \delta a^{ij} + \frac{P_i}{\Omega} \delta w^i + \frac{\Pi}{\Omega} \delta \Omega - \rho\, \delta \hat{B} - N^i \delta B_i \right)}$$

Galilean boost invariance requires $\delta_A S = 0$, and inserting $\delta_A a^{ij} = 0 \quad \delta_A \Omega = 0 \quad \delta_A \hat{B} = B_i A^i - \frac{1}{\Omega} \hat{\partial}_t \Lambda \quad \delta_A B_i = A_i - \partial_i \Lambda$

$$0 = \delta_A S = -\int dt\, \Omega \int d^d x \sqrt{a} \left(P_i A^i - \rho \left(B_i A^i - \frac{1}{\Omega} \hat{\partial}_t \Lambda \right) - N^i A_i + N^i \partial_i \Lambda \right)$$

$$= -\int dt\, \Omega \int d^d x \sqrt{a} \left(P_i - \rho B_i - N_i \right) A^i - \int dt \int d^d x \sqrt{a} \left(\rho\, \hat{\partial}_t \Lambda + \Omega N^i \partial_i \Lambda \right)$$

$\forall A^i, \Lambda$. This leads to

- conservation equation $\mathcal{C} = 0$
- $P_i = N_i + \rho B_i$ as anticipated

Further imposing $\delta_\xi S = 0$ will provide the two equations $\mathcal{E} = 0$ and $\mathcal{M}_i = 0$, which will explicitly involve the connection gauge field $(\hat{B}, B_i)$ in the form of $F = d\mathcal{B}$.

The reached dynamics is Galilean-boost invariant, however it contains F and is therefore different from the usual Galilean hydrodynamics discussed previously. Setting $F = 0$ is the option to recover the latter but this breaks Galilean boost since $\delta_A F \neq 0$.

we find indeed

$$
\begin{aligned}
\delta_\xi S = \int dt\, d^d x \Big\{ & \partial_t \Big[\sqrt{a}\, \big((\Pi - \rho \hat{B}) \xi^{\hat{t}} - (P_j + \rho B_j) \xi^{\hat{j}} \big) \Big] \\
& + \partial_i \Big[\sqrt{a}\, \Omega \Big(\xi^{\hat{t}} \big(\Pi^i - N^i \hat{B} + \frac{w^i}{\Omega} (\Pi - \rho \hat{B}) \big) \\
& \qquad - \xi^{\hat{j}} \big(\Pi^i{}_j + N^i B_j + w^i (P_j + \rho B_j) \big) \Big) \Big] \\
& + \int dt\, \Omega \int d^d x \sqrt{a} \Big\{ - \xi^{\hat{t}} \Big[\big(\tfrac{1}{\Omega} \hat{\partial}_t + \theta^w \big) \Pi + \Pi_{ij} \hat{\gamma}^{wij} - \hat{B} \big(\tfrac{1}{\Omega} \hat{\partial}_t + \theta^w \big) \rho \\
& \qquad + \underline{\hat{\nabla}_i (\Pi^i - N^i \hat{B})} + N^i \big(\tfrac{1}{\Omega} \hat{D} B_i + \hat{\gamma}^{wj}{}_i B_j \big) \Big] \\
& \quad + \xi^{\hat{j}} \Big[\big(\tfrac{1}{\Omega} \hat{D}_t + \theta^w \big) (P_j + \rho B_j) + \hat{\gamma}^{wi}{}_j (P_i + \rho B_i) \\
& \qquad + \hat{\nabla}^i (\Pi_{ij} + N_i B_j) - \rho \partial_j \hat{B} - N^i \hat{\nabla}_j B_i \Big] \Big\}
\end{aligned}
$$

where we have added and subtracted a boundary term which introduces a new momentum, Π_i, the heat flux.

Hence $\delta_\xi S = b.t. + \int dt\, \Omega \int d^d x \sqrt{a} \big\{ -\xi^{\hat{t}} \mathcal{E} + \xi^{\hat{j}} \mathcal{M}_j \big\}$

where, after some algebra (using in particular $p_{ij} = -\hat{\nabla}_{[i} B_{j]}$ and $\delta_i = \frac{1}{\Omega} \hat{D}_t B_i + \hat{\gamma}^{wj}{}_i B_j - \partial_i \hat{B}$) we find

$$
\boxed{
\begin{aligned}
\mathcal{E} &= -\hat{B}\, \mathcal{C} + \big(\tfrac{1}{\Omega} \hat{\partial}_t + \theta^w \big) \Pi + \Pi_{ij} \hat{\gamma}^{wij} + \hat{\nabla}_i \Pi^i + N^i \delta_i \\
\mathcal{M}_i &= B_i\, \mathcal{C} + \big(\tfrac{1}{\Omega} \hat{D}_t + \theta^w \big) P_i + P_j \hat{\gamma}^{wj}{}_i + \hat{\nabla}^j \Pi_{ij} + \rho \delta_i - 2 N^j p_{ji} \\
\mathcal{C} &= \big(\tfrac{1}{\Omega} \hat{\partial}_t + \theta^w \big) \rho + \hat{\nabla}_i N^i
\end{aligned}
}
$$

The boundary terms suggest the current

$$
\begin{aligned}
K &= \xi^{\hat{i}} (P_i + \rho B_i) - \xi^{\hat{t}} (\Pi - \rho \hat{B}) \\
K_i &= \xi^{\hat{j}} (\Pi_{ij} + N_i B_j) - \xi^{\hat{t}} (\Pi_i - N_i \hat{B})
\end{aligned}
$$

where $\xi = \xi^{\hat{t}} \frac{1}{\Omega} \hat{\partial}_t + \xi^{\hat{i}} \partial_i$ is a galilean Killing field, which furthermore leaves F invariant i.e. $\mathcal{L}_\xi \mathcal{B} = d\Lambda \Leftrightarrow \mathcal{L}_\xi F = 0$

We find the galilean divergence

$$\mathcal{K} = \left(\frac{1}{\Omega} \hat{\partial}_t + \theta^w\right) \kappa + \hat{\nabla}_i K^i$$

$$= P_i \left(\frac{1}{\Omega} \hat{D}_t \xi^{\hat{i}} - \hat{\gamma}^{w i}{}_j \xi^j\right) + \rho \frac{1}{\Omega} \hat{\partial}_t \Lambda + N^i \partial_i \Lambda$$

The last terms are irrelevant for the charge conservation ($Q_\kappa = \int_S d^dx \sqrt{a}\, \kappa$). The first however is an obstruction $P_i = 0$ or $\frac{1}{\Omega} \hat{D}_t \xi^{\hat{i}} - \hat{\gamma}^{w i}{}_j \xi^j = 0$

$$\Updownarrow$$

$$\mathcal{L}_\xi v = 0$$

This is an extra condition. Killings usually just obey $\mathcal{L}_\xi a^{ij} = 0$ $\mathcal{L}_\xi \tau = 0$